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ASYMPTOTIC DISTRIBUTIONS OF LINEAR AND NON-LINEAR COMBINATIONS OF EXTREME ORDER STATISTICS

ABDELHAKIM NECIR & BRAHIM BRAHIMI

Laboratory of Applied Mathematics, University of Biskra , P.O. Box 145 R.P.,
07000 Biskra, Algeria

ABSTRACT

The limit distributions of linear and non-linear combinations of the $k_n = o(n)$ order statistics of i.i.d. random variables whose maximum belongs to the domain of attraction of the Gumbel law are obtained. Our results may be applied in actuarial studies, estimation of scale-location parameters, estimation of squared deviation in tail of a distribution, robustness theory and detection of the outliers in statistical data. It is also closely related to the moment estimator of *Dekkers-Einmahl-de Hann* (1989) for the index of an extreme distribution. This study completes that of *Necir* (1990, 1991a, 1991b, 2000a, 2000b).

Key Words: Asymptotic normality, sums of extreme values, L-statistics, extreme order statistics, extreme squared deviation, quantile process, tails distributions.

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1. INTRODUCTION

Let $X_1, X_2, \ldots$, be a sequence of independent and identically distributed random variables with distribution function F. For each integer $n \geq 1$, let $X_{1,n} \leq \ldots \leq X_{n,n}$ denote the order statistics based on $X_1, \ldots, X_n$.
Assume that F belongs to the domain of attraction of the Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$, written $F \in D(\Lambda)$. This means that there exist constants $a_n > 0$ and b_n such that for all real x

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \lim_{n\to\infty} P(a_n^{-1}(X_{n,n} - b_n \leq x) = \Lambda(x) \quad (1.1)$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well known; see *Gnedenko* (1943), *de Hann* (1970), and *Galambos* (1987) Chapter 2. In particular, if (1.1) holds, then we may choose a_n and b_n by

$$a_n = U(1/en) - U(1/n) \text{ and } b_n = U(1/n), \quad (1.2)$$

where $U(1-s) = Q(s) = \inf\{x : F(s) \geq s\}$, $0 < s < 1$, is the quantile function pertaining to F, and e is the constant such that $\log e = 1$.
Let $\omega = \sup\{x : F(x) < 1\}$ denote the right endpoint of F.

Along this paper, we suppose that F satisfies von Mise's conditions (see e.g. *von Mises* (1936)) as follows :
(F) there exists an $x_0 < \omega$ such that F is twice continuously differentiable on (x_0,ω) with derivatives f and f', and

$$\lim_{x\uparrow\omega} \frac{f'(x)(1-F(x))}{f^2(x)} = -1. \quad (1.3)$$

Among distributions, which satisfy the Von Mises, conditions are the usual distributions as the *Exponential, Double-Exponential, Gamma, Logistic, Normal, Log-Normal, Gumbel, Weibull, Poisson* distributions.
Deheuvels, Haeusler and *Mason* (1990) has shown in proposition 1 that the conditions *(F)* are equivalents to the following
(U) there exist constants $0 < s_0 < 1$, $c > 0$ and a and a continuous function $b(.)$ with $b(v)\to 0$ as $v\downarrow 0$ such that

$$U(s) = a + \int_s^1 R(u)/u\, du, \; 0 < s < s_0, \quad (1.4)$$

where

$$R(u) = c\exp\left(\int_1^u b(v)/v\right). \quad (1.5)$$

Statements *(F)* and (U) are also equivalents if in *(F) f'* is the Radon-Nikodym derivative with respect to Lebesgue measure and in (U) $b(.)$ is a measurable function such the function $R(.)$ is well defined.)
It's clear, from (U) and representation (1.4) that the function U is differentiable on $0 < s < s_0$ and we have

$$-sU'(s) = R(s), \; 0 < s < s_0. \quad (1.6)$$

REMARK 1.2. Using (1.5) , it easy to check that the function $R(.)$ satisfies the following proprieties:

$$i)\ \lim_{s\to 0} R(\rho s)/R(s) = 1,$$

$$ii)\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)^d \frac{R(x_n)}{R(y_n)} = 0, \quad (1.7)$$

$$iii)\ R(1/n) \approx R(1/(n+1)) \text{ as } n\to\infty$$

for any $0 < \rho < \infty$, $0 < d < \infty$ and for any non-negative sequences (x_n) and (y_n) such that: $x_n \downarrow 0$, $y_n \downarrow 0$, and $x_n / y_n \downarrow 0$ as $n \to \infty$. On the other word the function $R(.)$ is slowly varying in the neighbourhood of zero.

REMARK 1.3. Under assumptions (U), (1.6)-(1.7) (*iii*) and the finite increments theorem imply that we also have

$$\lim_{n\to\infty} F^n(\hat{a}_n x + b_n) = \lim_{n\to\infty} P(\tilde{a}_n^{-1}(X_{n,n} - \tilde{b}_n) \le x) = \Lambda(x) \quad (1.8)$$

where

$$\tilde{a}_n = (n-1)^{-1} U'(1/(n+1)) \quad (1.9)$$

and

$$\tilde{b}_n = U(1/(n+1)).$$

Let J be positive measurable functions defined on [0.1] satisfy assumptions among *(H0 J* is bounded on [0.1].

(H1) J is uniformly Lipshitz of order $\alpha > 0$ there exists a $0 < M < \infty$ such that for $s,t \in [0.1], |J(s) - J(t)| \le M|s-t|^{\alpha}$

(H2) There exists a $0 < v < 1/2$, such that: $\int_0^1 s^{-1-2v} J(s) ds < \infty.$

(H3) There exists a $0 < \tau < 1/2$, such that: $\int_0^1 s^{-2\tau} J^2(s) ds < \infty.$

REMARK 1.4. Along this paper we use only assumptions *(H1)-(H2)* , while *(H3)* has been introduced in strong theorems given by Necir (2000a) (see also theorem A below).

Further, let $(k_n)_{n\ge 1}$ be an integer sequence satisfying, for suitable sequences p_n and q_n,

$$(K) \qquad 1 \le k_n \le n, k_n \approx p_n \uparrow \infty,$$

$k_n \approx q_n \downarrow 0$ as $n \to \infty$, where $u_n \approx v_n$ means that $u_n / v_n \to 1$ as $n \to \infty$.

Introducing a sequence of functions $\{J_n\}_{n\ge1}$ defined on [0,1] by

$$\begin{cases} J_n(t) = J(i/k_n) \text{ for } \dfrac{(i-1)}{k_n} < t \le \dfrac{i}{k_n}, \\ \qquad i = 1,\dots,k_n, \\ \qquad J_n(0) = J(1/k_n). \end{cases}$$

It's easy to verify, under *(H1)*, the sequence of functions $\{J_n\}_{n\ge1}$ is uniformly convergent on [0.1] to J, moreover, we have

$$\sup_{0\le s\le 1} |J_n(s) - J(s)| \le M k_n^{-\alpha} \quad (1.10)$$

and

$$Z = \sup_{0\le s\le 1} J_n(s) < \infty, \quad (1.11)$$

For each integer $n \ge 1$, and for any positive measurable function φ define on [0,1], let

$$\mu_{n,1}(\varphi) = k_n \int_0^1 U(k_n s/n)\varphi(s) ds, \quad (1.12)$$

$$\mu_{n,2}(\varphi) = k_n \int_0^1 U^2(k_n s/n)\varphi(s) ds, \quad (1.13)$$

$$v_{i,n} = n \int_{(i-1)/n}^{i/n} U(s) ds, \; i = 1,\dots,k_n; \quad (1.14)$$

and

$$\zeta_n(J) = \sum_{i=1}^{k_n} J(i/k_n) v_{i,n}^2 \quad (1.15)$$

We consider in this paper the statistics:

$$L_n(a) =: \sum_{i=1}^{k_n} a_{i,k_n} X_{n-i+1,n}, \text{ (L - Statistics Type)} \quad (1.16)$$

and

$$\tilde{D}_n(a) =: \sum_{i=1}^{k_n} a_{i,k_n} (X_{n-i+1,n} - v_{i,n})^2, \quad (1.17)$$

with

$$a_{i,k_n} = J(i/k_n); i = 1,\dots,k_n. \quad (1.18)$$

We also consider

$$L_n(b) =: \sum_{i=1}^{k_n} b_{i,k_n} X_{n-i+1,n}, \quad (1.19)$$

$$\tilde{D}_n(b) =: \sum_{i=1}^{k_n} b_{i,k_n} (X_{n-i+1,n} - v_{i,n})^2, \quad (1.20)$$

and

$$D_n(b) = \sum_{i=1}^{k_n} b_{i,k_n} (X_{n-i+1,n} - L_n(b))^2, \text{ (E}_\text{k}\text{ - Statistics type)} \quad (1.21)$$

with

$$b_{i,k_n} = \int_{(i-1)/k_n}^{i/k_n} J(s) ds; \; i = 1,\dots,k_n \quad (1.22)$$

The statistics $L_n(a)$ and $L_n(b)$ are very popular in *Nonparametric Estimate*, are well known by the "*L-statistics based upon extreme values*" (see e.g. *Shorack* and *Wellner* (1986), p. 660). These one are useful in estimation of scale-location parameters and detections of largest outliers in a sample of observations. They can be found in insurance statistics and extreme values theory. For instance, if X_1 ,X_2 ,...denote successive claims in an insurance business, one may seek the behavior of sums of the k_n extreme claims with a penalty function increasing with the claim size (see e.g. *Teugels* (1984) and *Beirlant* and *Teugels* (1987)). They can be also used to construct a robust estimator of the mean for a series of observations (see e.g. *Dixon* and *Tukey* (1968)). We can use these statistics to estimate the endpoints of distributions (see e.g. *Hall* (1982), *Csörgő* and *Mason* (1989) or *Falk* (1995)). We can also use the statistics $L_n(a)$ and $L_n(b)$ to improve the Hill (1975) estimator using the kernel estimate method (see *Deheuvels*, *Csörgő*, *Horváth* and *Mason* (1985)).

As for statistics $\widetilde{D}_n(b)$ and $D_n(b)$ represent the squared deviation between the largest order statistics and their expected values. They can be found in the area of estimation of the extreme index, for instance in a Pareto type situation one typically takes *log's* of the data to get back to the domain of attraction of the Gumbel distribution (see *Dekkers, Einmahl* and de *Hann* (1989) and *Tabbal* (1995)). They also can be used for the detection of outliers observations (see e.g. *Barnett* and *Lewis* (1994) p. 259, *Fung* and *Paul* (1985), *Tietjen* and *Moore* (1972), *Hawkins* (1979), *Dixon* and *Tukey* (1968)).

In the sequel, we shall see that there exists an algebraic relation between the three statistics $L_n(b)$, $\widetilde{D}_n(b)$ and $D_n(b)$. Then, the given of the asymptotic behaviors of the first and the second of these one gives also that of the third.

The smooth function J which defined above, will be suitably chosen according to the problem formulate. In general we chose the function J to obtain the asymptotic optimality of estimators (see, e.g. *Chernoff*, *Gastwirth* and *Johns* (1967), *Stigler* (1969, 1974), *Ruymgaart* and *van Zuijlen* (1977), *Mason* (1981), *Singh* (1981), *Mason* and *Shorack* (1990), *Shorack* and *Wellner* (1986); p. 640, *Csörgő*, *Deheuvels* and *Mason* (1985), and *Falk* (1995)). We also can chose the function J as the penalty function when X_1 ,X_2 ,... denote successive claims in an insurance business.

In the near future we shall present an application of our below results to improve Dekkers, Einmahl and de Hann's estimator in introducing a kernel function J. This idea was inspired from the results of *Deheuvels*, *Csörgő*, *Horváth* and *Mason* (1985) and that of Falk (1985).

Recently *Necir* (2000a) has described the almost sure behavior of statistic $\widetilde{D}_n(a)$ using the functional law of the iterated logarithm for the empirical quantile process (see, (3.1)) given by *Einmahl* and *Mason* (1988). Among these results is the following theorem.

THEOREM A (*Necir* (2000a)). Assume that *(F)* holds. Then for any sequence $\{k_n\}_{n\geq 1}$ satisfying (*K)* with $\log\ \log\ n = o(k_n^{2v})$, for a $0<v<1/2$, and for any function J satisfying (H1)-(H3), with probability one, there exists a constant $0 \leq l(J) \leq \int_0^1 s^{-1}J(s)ds$, 0 such that

$$\limsup_{n\to\infty}(\log\log n)^{-1}\left[R(k_n/n\right]^{-2}\times$$
$$\times\left\{\widetilde{D}_n(a)+\mu_{n,2}(J_n)-\zeta_n(J)\right\}=l(J)$$

and

$$\liminf_{n\to\infty}(\log\log n)^{-1}\left[R(k_n/n\right]^{-2}\times$$
$$\times\left\{\widetilde{D}_n(a)+\mu_{n,2}(J_n)-\zeta_n(J)\right\}=0$$

REMARK 1.4. To have the exact value of constant *l(J)* see the proposition given by *Necir* (2000a).
In this paper, we shall consider the corresponding limit distributions of statistics $\widetilde{D}_n(b)$ and $L_n(a)$. We profit for this study to describe moreover that of $D_n(b)$.

The general technique used along this paper was inspired from the famous results of *M.Csörgő S.Csörgő Horváth* and *Mason* (1986) concerning the asymptotic approximation of the uniform empirical quantile process (see lemma A in Section 3) by a sequence of Brownian bridges and those of *Csörgő*, *Deheuvels* and *Mason* (1985), *Lo* (1986), *Necir* (1990, 1991a, 1991b, 2000a, 2000b).

We shall show in the sequel, for suitable normalization' constants, that the limit distributions of $L_n(a)$, $L_n(b)$ and $D_n(b)$ are asymptotically standard normal $\mathcal{N}(0,1)$ as the statistics $\widetilde{D}_n(a)$ and $\widetilde{D}_n(b)$ has a particular limit distributions which will be precise later on.

Denote by $(W(t), 0 \leq t \leq 1)$ a standard Wiener process on [0,1]. To know more on such process consult Csörgő and Révész (1981).

2. MAIN RESULTS

TEHEOREM 1. Assume that *(F)* holds. Then for any sequence $\{k_n\}_{n\geq 0}$ satisfying *(K)* and for any function J satisfying (H1), we have

$$(k_n)^{-1/2}\left[R(k_n/n)\right]^{-1}\left\{L_n(a)-\mu_{n,1}(J_n)\right\}\xrightarrow{D}\int_0^1 s^{-1}J(s)W(s)ds$$

TEHEOREM 2. Assume that *(F)* holds. Then for any sequence $\{k_n\}_{n\geq 0}$ satisfying *(K)* and for any function J satisfying (H1) and (H2), we have

$$\left[R(k_n/n)\right]^{-2}\left\{\widetilde{D}_n(a)-\mu_{n,2}(J_n)-\zeta_n(J)\right\}\xrightarrow{D}\int_0^1 s^{-2}J(s)W^2(s)ds.$$

TEHEOREM 3. Assume that *(F)* holds. Then for any sequence $\{k_n\}_{n\geq 0}$ satisfying *(K)* and for any function J satisfying (H2), with $\int_0^1 J(s)ds = 1$, we have

$$(k_n)^{1/2}\left[R(k_n/n)\right]^{-2}\left\{D_n(a)-\mu_{n,2}(J)+\mu_{n,1}^2(J)\right\}$$
$$\xrightarrow{D}\int_0^1 s^{-1}\psi(s)W(s)ds,$$

where

$$\psi(s)=2(\log s-I(J))J(s),\quad 0<s<1, \tag{2.1}$$

with $I(J)=-\int_0^1 J(s)\log s ds.$

All constants $\mu_{n,1}(J_n)$, $\mu_{n,1}(J)$, $\mu_{n,2}(J_n)$, $\mu_{n,2}(J)$ and $\zeta_n(J)$ are defined in (1.12)-(1.15). $\left(\overset{D}{\rightarrow}\right)$ denotes convergence in distribution.

REMARK 2.1. The Wiener process introduced in last three theorems is define in the same probability space in which has defined the sequence if $X_1, X_2, \ldots$ of i.i.d. random variables (see lemma A in Section 3).
The following two corollaries 2.1 and 2.2 give us the exact limit distributions of statistics $L_n(a)$, and $D_n(b)$.

COROLLARY 2.1. Under assumptions of theorem 1, we have

$$(k_n)^{-1/2}\left[R(k_n/n)\right]^{-1}\left\{L_n(a)-\mu_{n,1}(J_n)\right\}\xrightarrow{D}N(0,\sigma^2(J)),$$

where

$$\sigma^2(J)=\int_0^1\int_0^1 s^{-1}t^{-1}\min(s,t)J(s)J(t)dsdt.$$

COROLLARY 2.2. Assume that (*F*) holds. For any sequence $\{k_n\}_{n\geq 0}$ satisfying *(K)* and for any function *J* satisfying (H2)

$$(k_n)^{1/2}\left[R(k_n/n)\right]^{-2}\left\{D_n(a)-\mu_{n,2}(J)+\mu_{n,1}^2(J)\right\}$$
$$\xrightarrow{D}N((0,\sigma^2(\psi)),$$

where

$$\sigma^2(\psi)=\int_0^1\int_0^1 s^{-1}t^{-1}\min(s,t)\psi(s)\psi(t)dsdt.$$

The following corollary shows that we also can obtain, relatively, the same result as theorem 1, whenever we take the weighting constants b_{i,k_n} instead of a_{i,k_n}.

COROLLARY 2.3. Assume that (*F*) holds. For any sequence $\{k_n\}_{n\geq 1}$ satisfying *(K)* and for any function *J* satisfying (H0).

$$(k_n)^{-1/2}\left[R(k_n/n)\right]^{-1}\left\{L_n(b)-\mu_{n,1}(J)\right\}\xrightarrow{D}\int_0^1 s^{-1}J(s)W(s)ds.$$

REMARK 2.2. It's clear from corollary 2.3, that the result of corollary 2.1 remains always valid for $L_n(b)$,
In proof of theorem 3, we shall see that the following corollary allow us to deduce the limiting distribution of statistic $D_n(b)$.

COROLLARY 2.4. Assume that (*F*) holds. For any sequence $\{k_n\}_{n\geq 1}$ satisfying *(K)* and for any function *J* satisfying (H2)

$$(k_n)^{1/2}\left[R(k_n/n)\right]^{-2}\left\{\widetilde{D}_n(b)+\mu_{n,2}(J)-\widetilde{\zeta}_n(J)\right\}$$
$$\xrightarrow{D}\int_0^1 s^{-2}J(s)W^2(s)ds.$$

where $\widetilde{\zeta}_n(J)=\sum_{i=1}^{k_n} b_{i,k_n}v_{i,n}{}^2.$

REMARK 2.3. The statistics $\widetilde{D}_n(a)$ and $\widetilde{D}_n(b)$ play an auxiliary roles in our study. Consequently we have interested only by there asymptotic bounds.

REMARK 2.4. By a simple integral calculation, it is easy to verify that from (H2), both of constants $\sigma^2(J)$ and $\sigma^2(\psi)$ are finites.

3. PRELIMINARY

Let $U_1, U_2, \ldots,$ be a sequence of independent uniform (0,1) random variables. For each integer $n\geq 1$, let $V_n(t)=U_{i,n}$, $(i-1)/n<t\leq i/n$, $i=1,\ldots,n$, with $V_n(0)=U_{1,n}$, where $U_{1,n}\leq\ldots\leq U_{n,n}$ are the order statistics based on $U_1, U_2, \ldots,$ be the sample quantile function.
We write the uniform quantile process as

$$\beta_n(s)=n^{1/2}\{V_n(s)-s\},\quad 0\leq s\leq 1 \tag{3.1}$$

We shall use the notation $\widetilde{\beta}_n(s)$ to denote the truncated uniform quantile process, which is equal to $\beta_n(s)$ for $1/(n+1)\leq s\leq n/(n+1)$ and defined to be 0 elsewhere.

The two sequence $\{X_n\}_{n\geq 1}$ and $\{Q(U_n)\}_{n\geq 1}$ are equal in distribution, and, consequently the two processes $\{X_{i,n}:1\leq i\leq n, n\geq 1\}$ and $\{Q(U_{i,n}):1\leq i\leq n, n\geq 1\}$ are equal in distribution as well. Therefore, without loss of generality, we may assume that $X_{i,n}=Q(U_{i,n})$ for all $1\leq i\leq n,$ and $n\geq 1$.

We begin by the following lemma which is the base of our results.

LEMMA A. (M. Csörgő, S. Csörgő, Horváth and Mason (1986)).

On a rich enough probability space carrying a sequence $U_1, U_2, \ldots,$ of independent uniform (0,1) random variables and a sequence of Brownian bridges $\{B_n(t) : 1 \le t \le n,\}_{n \ge 1}$ such that for all $0 < v < 1/2$ and for a large n,

$$\sup_{1/n \le s \le 1-1/n} \frac{|\beta_n(s) + B_n(s)|}{(s(1-s))^{1/2-v}} = O_p(n^{-v}) \tag{3.2}$$

On the sequel, we shall use the notation $\lambda_n =: k_n / n$.
The Taylor formula gives

$$n^{1/2}(U(1 - V_n(1 - \lambda_n u)) - U(1 - (1 - \lambda_n u)) =$$
$$= -\beta_n(1 - \lambda_n u) U'(\theta_n(\lambda_n u)), \tag{3.3}$$

where $\theta_n(z)$ is a function of $z \in [0,1]$ and $n \ge 1$, with values in the interval with endpoints z and $1 - V_n(1-z)$, defined via

$$U(1 - V_n(1-z)) - U(z) = -\{V_n(1-z) - 1 - z\} U'(\theta_n(z)), \tag{3.4}$$

Thus, we may write $\theta_n(\lambda_n s) = \gamma \lambda_n s + (1-\gamma)(1 - V_n(1 - \lambda_n s))$ for some appropriate $\gamma = \gamma_n(\lambda_n s) \in [0,1]$, depending upon $n \ge 1$ and $s \in [0,1]$. Observe now that for each integer $n>1$, we have $V(1 - i/n) = U_{n-i+1,n}$, and $U(1 - V(1 - i/n)) = Q(V(1 - i/n)) = X_{n-i+1,n}$ for $i=1,2,\ldots,n$.

4. PROOFS OF THE THEOREMS

4.1 Proof of theorem 1

Recalling that $\lambda_n =: k_n / n$. Using both of representations (3.2) and (3.3), we get

$$L_n(a) - \mu_{n,1}(J_n) = n\lambda_n \int_0^1 U(1 - V_n(1 - \lambda_n s)) J_n(s) ds$$
$$- n\lambda_n \int_0^1 U(\lambda_n s) J_n(s) ds = \sum_{i=1}^{5} L_{n,i},$$

where

$$L_{n,1} = n^{1/2} \lambda_n \int_{1/\lambda_n(n+1)}^{1} \beta_n(1 - \lambda_n s) \left(\frac{1 - U'(\theta_n(\lambda_n s))}{U'(\lambda_n s)} \right) \times$$
$$\times U'(\lambda_n s) J_n(s) ds,$$

$$L_{n,2} = n^{1/2} \lambda_n \int_{1/\lambda_n(n+1)}^{1} B_n(1 - \lambda_n s) \times U'(\lambda_n s)(J_n(s) - J(s)) ds,$$

$$L_{n,3} = -n^{1/2} \lambda_n \int_{1/\lambda_n(n+1)}^{1} (B_n(1 - \lambda_n s) + \beta_n(1 - \lambda_n s)) \times U'(\lambda_n s) J_n(s) ds,$$

$$L_{n,4} = n^{1/2} \lambda_n \int_{1/\lambda_n(n+1)}^{1} B_n(1 - \lambda_n s) \times U'(\lambda_n s) J(s) ds,$$

and

$$L_{n,5} = n\lambda_n R \int_0^{1/\lambda_n(n+1)} U(1 - V_n(1 - \lambda_n s)) J_n(s) ds$$
$$- n\lambda_n \int_0^{1/\lambda_n(n+1)} U(\lambda_n s) J_n(s) ds.$$

For the definition of o_p and O_p, which is used below, we refer to Serfling (1980) Section 1.2.5. Further, denote by ($\to_p$) convergence in probability.

We begin the proof of theorem 1 by the following.

LEMMA 4.1.1. Let *(K)*, *(F)* and (H1) be satisfied. Then $(n\lambda_n)^{-1/2} [R\lambda_n]^{-1} L_{n,2} \to_p 0$ as $n \to \infty$.

Proof. Set $\Theta_n(s) = [E|B_n(1 - \lambda_n s)|]$. By (1.10), we have

$$(n\lambda_n)^{-1/2} [R(\lambda_n)]^{-1} E|L_{n,2}| \le M n^{1/2} \lambda_n (n\lambda_n)^{-\alpha - 1/2} [R(\lambda_n)]^{-1}$$
$$\times \int_{1/\lambda_n(n+1)}^{1} \Theta_n(s) |U'(\lambda_n s)| ds \tag{4.1.1}$$

From (1.6), the right-hand-side of (4.1.1) is equal to

$$M(n\lambda_n)^{-\alpha} (\lambda_n) \int_{1/\lambda_n(n+1)}^{1} \Theta_n(s) s^{-1} [R(\lambda_n)]^{-1} R\lambda_n s ds \tag{4.1.2}$$

Using (1.7) *(i)*, expression (4.1.2) become as $n \to \infty$.

$$M(1 + o(1))(n\lambda_n)^{-\alpha} (\lambda_n)^{-1/2} \int_{1/\lambda_n(n+1)}^{1} \Theta_n(s) s^{-1} ds, \tag{4.1.3}$$

Since, for any $n \ge 1$ and $s \in [0,1]$,

$$E|B_n(1 - \lambda_n s)| \le (\lambda_n s)^{1/2} (1 - \lambda_n s)^{1/2} \le \lambda_n^{1/2} s^{1/2},$$

then, for a large n, expression (4.1.3) is less than or equal to

$$M(1 + o(1))(n\lambda_n)^{-\alpha} (\lambda_n)^{-1/2} \int_{1/\lambda_n(n+1)}^{1} (\lambda_n^{1/2} s^{1/2}) s^{-1} ds$$
$$\le M(1 + o(1))(n\lambda_n)^{-\alpha} (1/2)^{-1} = o(1),$$

(because $\alpha > 0$). This achieves the proof of lemma 4.1.1.□

LEMMA 4.1.2. Let *(K)*, *(F)* and (H1) be satisfied. Then $(n\lambda_n)^{-1/2} [R(\lambda_n)]^{-1} L_{n,3} \to_p 0$ as $n \to \infty$.

Proof. Let $0<\upsilon<1/2$ and setting:

$$\eta_{n,\upsilon}=\sup_{1/\lambda_n(n+1)\le s\le 1}\left[\frac{\left|\beta_n(1-\lambda_n s)+B_n(1-\lambda_n s)\right|}{((1-\lambda_n s)(\lambda_n s)))^{1/2-\upsilon}}\right], \quad (4.1.4)$$

It's easy to check, that

$$\eta_{n,\upsilon}=\sup_{1/\lambda_n(n+1)\le s\le 1}\left[\frac{\left|\widetilde{\beta}_n(1-\lambda_n s)+B_n(1-\lambda_n s)\right|}{((1-\lambda_n s)(\lambda_n s)))^{1/2-\upsilon}}\right], \quad (4.1.5)$$

consequently, by lemma A we have

$$\eta_{n,\upsilon}=O_p(n^{-\upsilon}),\ as\ n\to\infty, \quad (4.1.6)$$

Then, for n sufficiently large, using (1.11), (4.1.3), (1.6) and (1.7) *(i)* successively, we obtain

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}|L_{n,4}|\le$$

$$Zn^{1/2}\lambda_n\eta_{n,\upsilon}(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}\int_{1/\lambda_n(n+1)}^{1}((1-\lambda_n s)(\lambda_n s))^{1/2-\upsilon}$$

$$\times|U'(\lambda_n s)|ds,\le$$

$$Z\lambda_n\eta_{n,\upsilon}\int_0^1(\lambda_n s)^{1/2-\upsilon}(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}|U'(\lambda_n s)|ds$$

$$=O_p(n^{-v})\lambda_n^{-v}\int_0^1 s^{-1/2-v}[R(\lambda_n)]^{-1}[R(\lambda_n s)]ds$$

$$=O_p((n\lambda_n))^{-v},$$

which converges to zero as $n\to\infty$, because $n\lambda_n\uparrow\infty$ as $n\uparrow\infty$, with $0<v<1/2$.□

LEMMA 4.1.3. Let *(K)*, *(F)* and (H1) be satisfied. Then $(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}L_{n,5}\to_p 0$ as $n\to\infty$.

Proof. We have,

$$U(1-V_n(1-\lambda_n s))=X_{n,n},$$

and

$$J_n(s)=J(1/n\lambda_n)\ \text{for}\ 1/\lambda_n(n+1)\le s\le 1,$$

consequently,

$$L_{n,5}=n\lambda_n J(1/n\lambda_n)X_{n,n}(\lambda_n(n+1))^{-1}$$

$$-n\lambda_n J(1/n\lambda_n)\int_0^{1/\lambda_n(n+1)}U(\lambda_n s)ds.$$

$$=J(1/n\lambda_n)\left\{n(n+1)^{-1}X_{n,n}-n\lambda_n\int_0^{1/\lambda_n(n+1)}U(\lambda_n s)ds\right\}.$$

Combining (1.8) with (1.9), we get

$$L_{n,5}=:\overline{L}_{n,5}+\hat{L}_{n,5},$$

where

$$\overline{L}_{n,5}=nJ\left(\frac{1}{n\lambda_n}\right)(n+1)^{-1}\times O_p(1)(n+1)^{-1}U'\left(\frac{1}{n+1}\right)$$

and

$$\hat{L}_{n,5}=nJ\left(\frac{1}{n\lambda_n}\right)\left\{(n+1)^{-1}U\left(\frac{1}{n+1}\right)-\int_0^{1/(n+1)}U(s)ds\right\}.$$

Using (1.6) \$ and (1.7) *(iii)*, for a large n, we get

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}\overline{L}_{n,5}=O_p(1)(1+o(1))J(1/n\lambda_n)$$
$$\times\left[(n\lambda_n)^{-1/2}R(1/n)/R(\lambda_n)\right].$$

Since $n\lambda_n\uparrow\infty$ and $\lambda_n\downarrow 0$ as $n\to\infty$, then using (1.7) *(ii)*, we get

$$(n\lambda_n)^{-1/2}R(1/n)R(\lambda_n)=o(1)\ \text{as}\ n\uparrow\infty,$$

therefore

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}\overline{L}_{n,5}=o_p(1)\ \text{as}\ n\uparrow\infty,$$

An integration by part gives

$$\int_0^{1/(n+1)}U(s)ds=(n+1)^{-1}U(1/(n+1))-\int_0^{1/(n+1)}sU'(s)ds.$$

Then, substituting (4.1.4) in to $\hat{L}_{n,5}$ and using (1.6) and (1.7) *(iii)*, yields for a large n

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}\hat{L}_{n,5}$$

$$=-(1+o(1))J(1/n\lambda_n)\times\left[(n\lambda_n)^{-1/2}R(1/n)/R(\lambda_n)\right]$$

which converges to zero as $n\to\infty$, by (1.7) *(ii)*.□

LEMMA 4.1.4. Let *(K)*, *(F)* and (H1) be satisfied. Then, as $n\to\infty$.

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}L_{n,4}\overset{D}{\to}\int_0^1 J(s)W(s)ds.$$

Proof. By a same argument as above we can write for a large n

$$(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}L_{n,4}$$

$$=-\lambda_n^{-1/2}\int_{1/\lambda_n(n+1)}^{1}B_n(1-\lambda_n s)[R(\lambda_n)]^{-1}[R(\lambda_n s)]s^{-1}J(s)ds \quad (4.1.8)$$

$$=-(1+o(1))\lambda_n^{-1/2}\int_{1/\lambda_n(n+1)}^{1}B_n(1-\lambda_n s)s^{-1}J(s)ds \quad (4.1.9)$$

Since $-B_n(t)\overset{D}{=}N(0,t(1-t))$ for every $n\geq 1$, consequently we can put

$$-B_n(t)\overset{D}{=}B(t),\ 0\leq t\leq 1,\ \text{ for } n=1,2,..., \qquad (4.1.10)$$

when $B(t)$, $0\leq t\leq 1$, is a Brownian bridge define on the same probability space. ($\overset{D}{=}$ denotes equality in distribution).
Then the right-hand-side of (4.1.5), without loss of generality, can be written as follows

$$(1+o(1))\lambda_n^{-1/2}\int_{1/\lambda_n(n+1)}^{1}B(1-\lambda_n s)s^{-1}J(s)ds. \qquad (4.1.11)$$

On the other hand, from the proprieties of the Brownian bridges and Wiener processes we have

$$(i)\ \ B(t)\overset{D}{=}W(t)-tW(1),\ 0\leq t\leq 1, \qquad (4.1.12)$$

$$(ii)\ s^{-1/2}W(st)\overset{D}{=}W(t), \text{ for any } 0\leq t<\infty, s>0, \qquad (4.1.13)$$

$$(iii)\ \ B(t)\overset{D}{=}B(1-t),\ 0\leq t\leq 1. \qquad (4.1.14)$$

Then using (4.1.8) (4.1.10), expression (4.1.7) is equal in distribution to

$$(1+o(1))\int_{1/\lambda_n(n+1)}^{1}W(s)s^{-1}J(s)ds-$$

$$-(1+o(1))\lambda_n^{1/2}W(1)\int_{1/\lambda_n(n+1)}^{1}s^{-1}J(s)ds.$$

Since $W(1)\overset{D}{=}N(0,1)$, then the second term of last expression converges in probability to zero as $n\to\infty$, which achieves proof of this lemma. □
Recalling that

$$\theta_n(\lambda_n s)=\gamma\lambda_n s+(1-\gamma)(1-V_n(1-\lambda_n s)),$$

for some appropriate $\gamma=\gamma_n(\lambda_n s)\in[0,1]$ depending upon $n\geq 1$ and $s\in[0,\lambda_n]$, or

$$\theta_n(s)=(1-\delta)s+\delta(1-V_n(1-s)),$$

or some appropriate $\delta=\delta_n(s)\in[0,1]$ depending upon $n\geq 1$ and $s\in[0,\lambda_n]$.
The following lemma gives some results with respect to asymptotic behavior of $\theta_n(s)$.

LEMMA 4.1.5. Let *(F)* be satisfied. For a large *n*, we have for all $1/(n+1)\leq s\leq\lambda_n$,

$$U'\left(\frac{\theta_n(s)}{U'(s)}\right)=\begin{cases}(s/\theta_n(s))^{1+\varepsilon}, \text{if } s<\theta_n(s)<1-V_n(1-s)\\ (s/\theta_n(s))^{1-\varepsilon}, \text{if } 1-V_n(1-s)<\theta_n(s)<s\end{cases}$$

for any $0<\varepsilon<1$.

Proof. Let $1/(n+1)\leq s\leq\lambda_n$. We have $\theta_n(s)=(1-\delta)s+\delta(1-V_n(1-s))$, from of Glinvenko-Cantelli' theorem, we have, almost surely, for a large *n*

$$\sup_{0<s<1}\left|1-s-V_n(1-s)\right|=o(1),$$

it follows that, almost surely, for a large *n,* $\theta_n(s)=s+o(1)$, and consequently, both of s and $\theta_n(s)$ are in right neighbourhood of zero.
Suppose that $s<\theta_n(s)<1-V_n(1-s)$. Let $0<\varepsilon<1$. By (1.5), it's easy to verify that for a large *n* we have $R(\theta_n(s))/R(s)=(s/\theta_n(s))^{-\varepsilon}$, for $1/(n+1)\leq s\leq\lambda_n$.
Suppose now that $1-V_n(1-s)<\theta_n(s)<s$. By a same arguments as last, we prove that, for a large *n,* $R(\theta_n(s))/R(s)=(s/\theta_n(s))^{\varepsilon}$, for $1/(n+1)\leq s\leq\lambda_n$.
This achieves proof of this lemma.□

LEMMA 4.1.6. Let *(K)* and *(F)* be satisfied. Then, we have almost surely

$$\sup_{1/\lambda_n(n+1)<s<1}\left(\frac{1-U'\theta_n(\lambda_n s)}{U'(\lambda_n s)}\right)=o(1), \text{ as } n\to\infty.$$

Proof. Let $0<\varepsilon<1$. From (1.4) and (1.5) we can easel show, that for a large *n*

$$\left(\frac{1-U'(\theta_n s)}{U'(s)}\right)=(s/\theta_n(s))^{1\pm\varepsilon}.$$

In view of the Theorem of Hàjek and Bickel (1972) (see e.g. Shorack and Wellner (1986), p. 640), we have almost surely for a large *n* $\sup_{0<s<1}\left|1-s/\theta_n(s)\right|=o(1)$. Since $1\pm\varepsilon>0$, then with probability 1 a $n\to\infty$,

$$\sup_{1/(n+1)<s<\lambda_n}\left(1-\frac{s}{\theta_n(s)}\right)^{1\pm\varepsilon}=o(1),$$

which achieves proof of lemma 4.1.6.□

LEMMA 4.1.7. Let *(K)*, *(F)* and (H1) be satisfied. Then $(n\lambda_n)^{-1/2}[R(\lambda_n)]^{-1}L_{n,1}\to_p 0$ as $n\to\infty$.

Proof. In view of lemma 4.1.6, for a large *n,* we have

$$L_{n,1} = o_p(1) n^{1/2} \lambda_n \int_{1/\lambda_n(n+1)}^{1} \beta_n (1-\lambda_n s) U'(\lambda_n s) J_n(s) ds,$$
$$= o_p(1)\left[L_{n,2} + L_{n,3} + L_{n,4} + L_{n,5}\right]$$

It's now clear, that the proof of this lemma achieves by applying successively lemmas 4.1.1-4.1.6. □

4.2 Proof of theorem 2

First recall that $k_n \int_0^{1/k_n} J_n(s) ds = J(1/k_n)$. Write

$$\widetilde{D}_n(b) + \mu_{n,2}(J_n) - \zeta_n(J) =: A_n^{'} + A_n^{''} + A_n^{'''} \qquad (4.2.1)$$

where

$$A_n^{'} = J(1/k_n)(X_{n,n} - v_{1,n})^2$$

$$A_n^{''} = \sum_{i=2}^{k_n} J(i/k_n) X_{n-i+1,n}^2 - 2\sum_{i=2}^{k_n} J(i/k_n) v_{i,n} X_{n-i+1,n}$$
$$+ k_n \int_0^{1/k_n} U^2(k_n s/n) J_n(s) ds,$$

$$A_n^{'''} = k_n \int_0^{1/k_n} U^2(k_n s/n) J_n(s) ds - J(1/k_n) v_{1,n}^2.$$

Recalling that $\mu_{n,2}(J)$, $v_{i,n}$ and $\zeta_n(J)$ are constants as in 1.14 and 1.15 respectively.
We shall show in lemma 4.2.1 and 4.2.2 that for a large n

$$\left[R(\lambda_n)\right]^{-2} A_n^{'} = \left[R(\lambda_n)\right]^{-2} A_n^{''} = o_p(1).$$

LEMMA 4.2.1. Let *(K)*, *(F),* (H1) and (H2) be satisfied. Then

$$\left[R(\lambda_n)\right]^{-2} A_n^{'} \to_p 0 \text{ as } n \to \infty.$$

Proof. Recalling that $v_{1,n}(J) = n\int_0^{1/n} U(s) ds.$ From 4.1.4, 1.8 and 1.9 and by a same argument as proof of lemma 4.1.3, we write, for a large n

$$A_n^{'} = J(1/\lambda_n)\left[O_p(1) n^{-1} U'(1/n) + n\int_0^{1/n} s U'(s) ds\right]^2$$

$$J(1/\lambda_n)(R(1/n) + (1+o(1) R(1/n)))^2. \qquad (4.2.2)$$

Under (H2) we have

$$k_n \int_0^{1/k_n} J(s) ds \le k_n^{-2v} \int_0^{1/k_n} s^{-1+2v} J(s) ds$$
$$= o\left(k_n^{-2v}\right), \text{ as } n \to \infty,$$

which implies from (1.10) that

$$J(1/n\lambda_n) = O(k_n^{-\alpha}) + k_n \int_0^{1/k_n} J(s) ds$$
$$= O(k_n^{-\alpha}) + o\left(k_n^{-2v}\right), \text{ as } n \to \infty.$$

Consequently
$$(R(k_n/n))^{-2} A_n^{'} =$$
$$(1+o(1))\left[\left(\frac{R(1/n)}{R(k_n/n)}\right)^2 (O(1)(k_n^{-\alpha}) + o(1)\left(k_n^{-2v}\right))\right]$$

hence for a large n, $(R(k_n/n))^{-2} A_n^{'} = o(1)$, from (1.7) *(ii)*. □
For each $n \ge 1$, set

$$\Omega_n(s) =: \beta_n(1-\lambda_n s) + B_n(1-\lambda_n s), \; 1/\lambda_n(n+1) < s < 1.$$

We have

$$A_n^{''} = n\lambda_n \int_{1/n\lambda_n(n+1)}^{1} \left(U(1 - V_n(1-\lambda_n s) - U(\lambda_n s)^2\right) J_n(s) ds$$
$$=: \varepsilon_n + T_n, \qquad (4.2.3)$$

where

$$\varepsilon_n = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} \beta_n^2 (1+\lambda_n s)$$
$$\times \left\{1 - (U'(\theta_n(\lambda_n s))/U'(\lambda_n s))^2\right\} U'^2(\lambda_n s) J_n(s) ds,$$

$$T_n = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} \beta_n^2 (1+\lambda_n s) U'^2(\lambda_n s) J_n(s) ds.$$

Remark now that T_n can be written as follows

$$T_n =: T_{n1} + T_{n2} + T_{n3} + T_{n4}, \qquad (4.2.4)$$

where

$$T_{n1} = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} \Omega_n^2 (1+\lambda_n s) U'^2(\lambda_n s) J_n(s) ds.$$

$$T_{n2} = -2\lambda_n \int_{1/n\lambda_n(n+1)}^{1} \Omega_n(s) B_n(1+\lambda_n s) U'^2(\lambda_n s) J_n(s) ds.$$

$$T_{n3} = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} B_n^2 (1+\lambda_n s) U'^2(\lambda_n s)(J_n(s) - J(s)) ds,$$

and

$$T_{n4} = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} B_n^2 (1+\lambda_n s) U'^2(\lambda_n s) J(s) ds.$$

We have

$$T_{n4} \overset{D}{=} \lambda_n \int_{1/n\lambda_n(n+1)}^{1} B_n^2 (\lambda_n s) U'^2(\lambda_n s) J(s) ds.$$

On the other hand we have

$$B_n^2(\lambda_n s) \stackrel{D}{=} (W(\lambda_n s) - \lambda_n s W(1))^2$$
$$= W^2(\lambda_n s) - 2\lambda_n s W(1) s W(\lambda_n s) + \lambda_n^2 s^2 W^2(1).$$

Consequently T_{n4} can be also written as follows

$$T_{n4} =: T_{n4}^{'} + T_{n4}^{''} + T_{n4}^{'''},$$

where

$$T_{n4}^{'} = \lambda_n \int_{1/n\lambda_n(n+1)}^{1} W^2(\lambda_n s) U'^2(\lambda_n s) J(s) ds,$$

$$T_{n4}^{''} = 2\lambda_n^2 W(1) \int_{1/n\lambda_n(n+1)}^{1} W(\lambda_n s) U'^2(\lambda_n s) J(s) ds,$$

and

$$T_{n4}^{'''} = \lambda_n^3 W^2(1) \int_{1/n\lambda_n(n+1)}^{1} s^2 U'^2(\lambda_n s) J(s) ds.$$

The below two lemmas give us the asymptotic behaviors of terms $T_{n4}^{''}$, $T_{n4}^{'''}$ and $T_{n4}^{'}$.

LEMMA 4.2.2. Let *(K)*, *(F)* and (H1) (H2) be satisfied. Then as $n \to \infty$

$$[R(\lambda_n)]^{-2} T_{n4}^{''} = [R(\lambda_n)]^{-2} T_{n4}^{'''} = o_p(1).$$

Proof. First we have $W(1) = O_p(1)$, moreover we have $|W(\lambda_n s)| \le (\lambda_n s)^{1/2}$, therefore, from (1.7) *(ii)*,

$$[R(\lambda_n)]^{-2} E\left|T_{n4}^{''}\right| = 2O_p(1)(1+o(1))\lambda_n^{1/2} \int_0^1 s^{-3/2} J(s) ds,$$

which converges, in probability, to zero as $n \to \infty$, since $\lambda_n \to 0$ and $\int_0^1 s^{-3/2} J(s) ds < \infty$. On the other hand we have from (1.6) and (1.7) *(i)*

$$[R(\lambda_n)]^{-2} T_{n4}^{'''} = \lambda_n O_p(1)(1+o(1)) \int_0^1 J(s) ds,$$

which converges, in probability, to zero as $n \to \infty$ as well. □

LEMMA 4.2.3. Let *(K)*, *(F)* and (H1)-(H2) be satisfied. Then

$$[R(\lambda_n)]^{-2} T_{n4}^{'} \stackrel{D}{\to} \int_0^1 J(s) W^2(s) ds, \text{ as } n \to \infty.$$

Proof. It suffices to apply (1.6), (1.7) *(ii)* and (4.1.6) together.

Recapitulate, the two last lemmas show that T_{n4} is the only term in series (4.2.4) which gives us the limit distribution as in theorem 2. Hence, in order to achieve the proof of theorem 2, it suffices to show that for a large *n*

$$[R(\lambda_n)]^{-2} T_{n1} = [R(\lambda_n)]^{-2} T_{n2}$$
$$= [R(\lambda_n)]^{-2} T_{n3}$$
$$= [R(\lambda_n)]^{-2} \varepsilon_n = o_p(1),$$

which will be the aim of the following lemmas.

LEMMA 4.2.4. Let *(K)*, *(F)*, (H1) and (H2) be satisfied. Then

$$[R(\lambda_n)]^{-2} T_{n1} = o_p(1) \text{ as } n \to \infty.$$

Proof. First observe, by (1.10) we have

$$J_n(s) = O((n\lambda_n)^{-\alpha}) + J(s), \text{ for any } 0 \le s \le 1, \tag{4.2.5}$$

then from (1.6) and (1.7) (ii) we can write

$$[R(\lambda_n)]^{-1} T_{n1} = (1+o(1))(T_{n1}^{'} + T_{n1}^{''}),$$

where

$$T_{n1}^{'} = O((n\lambda_n)^{-\alpha}) \times$$
$$\lambda_n^{-1} \int_{1/n\lambda_n(n+1)}^{1} (\beta_n(1-\lambda_n s) + B_n(1-\lambda_n s))^2 s^{-2} ds.$$

and

$$T_{n1}^{''} = \lambda_n^{-1} \int_{1/n\lambda_n(n+1)}^{1} (\beta_n(1-\lambda_n s) + B_n(1-\lambda_n s))^2 s^{-2} ds.$$

We have

$$0 \le T_{n1}^{'} = O((n\lambda_n)^{-\alpha}) \lambda_n^{-1} (\eta_{n,v})^2 \int_{1/n\lambda_n(n+1)}^{1} (\lambda_n s)^{1-2v} s^{-2} ds$$
$$= O((n\lambda_n)^{-\alpha}) O_p(n^{-2v}) \lambda_n^{-2v} \int_{1/n\lambda_n(n+1)}^{1} s^{-1-2v} ds$$
$$= O((n\lambda_n)^{-2v})(-2v)^{-1}(1-n\lambda_n)^{2v}) O((n\lambda_n)^{-\alpha})$$
$$= O_p((n\lambda_n)^{-2v-\alpha}) - O_p(1) O((n\lambda_n)^{-\alpha}). \tag{4.2.6}$$

Since $2v > 0$ and $\alpha > 0$, with $\lambda_n \to 0$ and $n\lambda_n \to 0$ as $n \to \infty$, then (4.2.6) converges, in probability, to zero as $n \to \infty$. By a same technique, we also show, under (H2), that

$$0 \le T_{n1}^{''} = O_p(n^{-v}) \lambda_n^{1-2v} \int_0^1 s^{-1-2v} J(s) ds,$$

which converges to zero as $n \to \infty$. □

LEMMA 4.2.5. Let *(K)*, *(F)*, (H1) and (H2) be satisfied. Then

$$[R(\lambda_n)]^{-2} T_{n2} = o_p(1) \text{ as } n \to \infty.$$

Proof. We have

$$[R(\lambda_n)]^{-2} T_{n2} = -(1+o(1))\{T_{n2}^{'} + T_{n2}^{''}\},$$

where

$$T_{n2}^{'} = O_p(n^{-\nu})O((n\lambda_n)^{-\alpha}) \times$$
$$\times 2\lambda_n^{-1}\int_{1/\lambda_n(n+1)}^{1}(\lambda_n s)^{\nu}(1-\lambda_n s)^{\nu}s^{-2}B_n(1-\lambda_n s)ds,$$

and

$$T_{n2}^{''} = O_p(n^{-\nu}) \times$$
$$\times 2\lambda_n^{-1}\int_{1/\lambda_n(n+1)}^{1}(\lambda_n s)^{\nu}(1-\lambda_n s)^{\nu}s^{-2}B_n(1-\lambda_n s)J(s)ds.$$

Then using a same technics as proofs of lemmas 4.1.1. and 4.2.4, we show easily, under assumptions that for a large n, $T_{n2}^{'} = T_{n2}^{''} = o_p(1)$, consequently the details are omitted.□

LEMMA 4.2.6. Let *(K)*, *(F)*, (H1) and (H2) be satisfied. Then

$$[R(\lambda_n)]^{-2}T_{n3} = o_p(1) \text{ as } n \to \infty.$$

Proof. Obvious, from (1.10) and the fact that for any $n \geq 1$ and $s \in [0,1]$,

$E|B_n(1-\lambda_n s)| \leq (\lambda_n s)^{1/2}(1-\lambda_n s)^{1/2} \leq \lambda_n^{1/2}s^{1/2}$. □

Finally we achieve proof of theorem 1 by the following lemma which shows the asymptotic behavior of term ε_n appearing in (4.2.3).

LEMMA 4.2.7. Let *(K)*, *(F)*, (H1) and (H2) be satisfied. Then

$$[R(\lambda_n)]^{-2}\varepsilon_n = o_p(1) \text{ as } n \to \infty.$$

Proof. It's straightforward, by applying successively lemmas, (4.1.6), (4.2.2)-(4.2.7). □

We finish proof of theorem 2 by the following lemma.

LEMMA 4.2.8. Let *(K)*, *(F)*, (H1) and (H2) be satisfied. Then

$$[R(\lambda_n)]^{-2}A_n^{'''} = o_p(1) \text{ as } n \to \infty.$$

Proof. Recall that $v_{1,n} = n\int_0^{1/n}U(s)ds.$ Since $J_n(s) = J(1/k_n)$, on $0 \leq s \leq 1/k_n$, then by a change of variables we obtain

$$A_n^{'''} = J(1/k_n)\left[n\int_0^{1/n}U^2(s)ds. - v_{1,n}^2\right] =: J(1/k_n)S_n.$$

An integration by part gives

$$n\int_0^{1/n}U^2(s)ds = U^2(1/n) - 2n\int_0^{1/n}sU'(s)U(s)ds,$$

and

$$v_{1,n} = U(1/n) - n\int_0^{1/n}sU'(s)ds.$$

Therefore

$$S_n =: S_{n1} + S_{n2},$$

where

$$S_{n1} = -\left(n\int_0^{1/n}sU'(s)ds\right)^2, \text{ and}$$

$$S_{n2} = -2n\int_0^{1/n}sU'(s)(U(s)-U(1/n))ds$$

It's clear that, From (1.6) and (1.7) *(i)*, we have

$$S_{n1} = S_{n2} = (1+o(1))(R(1/n))^2, \text{ as } n \to \infty.$$

On the other hand, remark that

$$J(1/k_n) = k_n\int_0^{1/k_n}J_n(s)ds = O(k_n^{-\alpha}) + k_n\int_0^{1/k_n}J_n(s)ds.$$

From (H3) we have

$k_n\int_0^{1/k_n}J_n(s)ds \leq k_n^{-2\nu}\int_0^{1/k_n}s^{-1+2\nu}J_n(s)ds = o(k_n^{-2\nu}).$ as $n \to \infty$. Consequently

$$(R(k_n/n))^{-2}S_{n1} = (R(k_n/n))^{-2}S_{n2}$$
$$= (1+o(1))\left(\frac{R(1/n)}{R(k_n/n)}\right)^2\left\{O(1)k_n^{-\alpha} + o(1)k_n^{-2\nu}\right\}.$$

Finally, in applying (1.7) *(ii)* we achieve this lemma. □

4.3 PROOF OF THEOREM 3

The general idea of proof of the present theorem, consists to represent the statistics $D_n(b)$ on function of $L_n(b)$ and $\widetilde{D}_n(b)$. Recall that $\int_0^1 J(s)ds = 1$. First, we can verify that

$$D_n(b) =: \pi_{n1} + \pi_{n2} + \pi_{n3} + \pi_{n4} \tag{4.3.1}$$

where

$$\pi_{n1} = \int_0^1 J(s)[U(1-V_n(1-\lambda_n s) - U(\lambda_n s)]^2 ds,$$

$$\pi_{n2} = 2\int_0^1 J(s)U(\lambda_n s)[U(1-V_n(1-\lambda_n s) - U(\lambda_n s)]^2 ds,$$

$$\pi_{n3} = -L_n^2(b)\int_0^1 J(s)ds = -L_n^2(b),$$

$$\pi_{n4} = \int_0^1 J(s)U^2(\lambda_n s)ds.$$

Setting $K(s) = \int_0^s J(t)dt.$ It's clear that $K(0) = 0$ and $K(1) = 1$. Moreover, we show easily, from (H2), that $\lim_{s \downarrow 0} s^{-1}K(s) < \infty.$ Recall that

$\overline{\mu}(J)=\int_0^1 J(s)U(\lambda_n s)ds.$ An integration by part gives

$$\overline{\mu}(J)=U(\lambda_n)-\lambda_n\int_0^1 K(s)U'(\lambda_n s)ds.$$

(we have used the fact that $\lim_{t\downarrow 0} tU(t)=0$).
Substituting last result in to term

$$\Gamma_{n2}=-2U(\lambda_n)(L_n(b)-\overline{\mu}(J))+2\lambda_n\int_0^1 K(s)U'(\lambda_n s)ds$$
$$\times(L_n(b)-\overline{\mu}(J))$$
$$=-2U(\lambda_n)\int_0^1 J(s)\left[U(1-V_n(1-\lambda_n s))-U(\lambda_n s)\right]ds$$
$$+2\lambda_n\int_0^1 K(s)U'(\lambda_n s)ds\times(L_n(b)-\overline{\mu}(J))$$

$$D_n(b)=:\Delta_{n1}+\Delta_{n2}+\Delta_{n3}+\Delta_{n4}+\Delta_{n5} \tag{4.3.2}$$

where

$$\Delta_{n1}=\int_0^1 J(s)\left[U(1-V_n(1-\lambda_n s))-U(\lambda_n s)\right]^2 ds,$$

$$\Delta_{n2}=2\int_0^1 J(s)\left[U(1-V_n(1-\lambda_n s))-U(\lambda_n s)\right]$$
$$\times(U(\lambda_n s)-U(\lambda_n))ds,$$

$$\Delta_{n3}=-(L_n(b)-\overline{\mu}(J))^2,$$

$$\Delta_{n4}=-2\lambda_n\int_0^1 K(s)U'(\lambda_n s)ds(L_n(b)-\overline{\mu}(J)),$$

and

$$\Delta_{n5}=\int_0^1 J(s)U^2(\lambda_n s)ds-(\overline{\mu}(J))^2.$$

We show in the sequel that for a large *n*

$$(n\lambda_n)^{1/2}R(\lambda_n)^{-2}\Delta_{ni}=o_p(1),\ i=1,3;$$

while

$$(n\lambda_n)^{1/2}R(\lambda_n)^{-2}\Delta_{n2}\xrightarrow{D}2\int_0^1 s^{-1}J(s)\log sW(s)ds$$

as $n\to\infty$ and

$$(n\lambda_n)^{1/2}R(\lambda_n)^{-2}\Delta_{n4}\xrightarrow{D}-2I(J)\int_0^1 s^{-1}J(s)W(s)ds,$$ where

$$I(J)=-\int_0^1 J(s)\log s ds.$$

LEMMA 4.3.1. Let *(K)*, *(F)* and (H2) be satisfied. Then

$$(n\lambda_n)^{1/2}R(\lambda_n)^{-2}\Delta_{n1}=o_p(1)\ \text{ as }\ n\to\infty.$$

Proof. Expanding statistic $\widetilde{D}_n(b)$, we show that

$$\Delta_{n1}=(n\lambda_n)^{-1}(\widetilde{D}_n(b)+\mu_{n,2}(J)-\zeta_n(J)).$$

Then from theorem we have for a large *n*
$(R(\lambda_n))^{-2}(\widetilde{D}_n(b)+\mu_{n,2}(J)-\zeta_n(J))=O_p(1),$
Consequently

$$(n\lambda_n)^{-1/2}(R(\lambda_n))^{-2}(\widetilde{D}_n(b)+\mu_{n,2}(J)-\zeta_n(J))$$
$$=O_p((n\lambda_n)^{-1/2}).$$

which converges to zero as $n\to\infty$. This achieves proof of the present lemma. □

LEMMA 4.3.2. Let *(K)*, *(F)* and (H2) be satisfied. Then as $n\to\infty$.

$$(n\lambda_n)^{1/2}R(\lambda_n)^{-2}\Delta_{n2}\xrightarrow{D}-2\int_0^1 s^{-1}J(s)\log sW(s)ds.$$

Proof. From the finite increments theorem, there exits a function

$$\frac{U(\lambda_n s)-U(\lambda_n)}{\log(\lambda_n s)-\log\lambda_n}=\frac{U'(\rho_n(s))}{(\log(\rho_n(s)))'}.$$

Moreover the right- hand-side of last expression is equal , by (1.16), to

$$-(1+o(1))R(\rho_n(s)),\ \text{ as }\ n\to\infty,$$

consequently as $n\to\infty$

$$U(\lambda_n s)-U(\lambda_n)=-R(\rho_n(s))\log s \tag{4.3.3}$$

Further it is easy to verify that we have also , from (1.7) *(ii)*,

$$R(\rho_n(s))=(1+o(1))R(\lambda_n) \tag{4.3.4}$$

Substituting (4.3.3) and (4.3.4) in to Δ_{n2}, we get

$$\Delta_{n2}=-2(1+o(1))R(\lambda_n)\int_0^1 J(s)\log s$$
$$\times\left[U(1-V_n(1-\lambda_n s))-U(\lambda_n s)\right]ds$$
$$=2(1+o(1))R(\lambda_n)\left[L_n(c)-\overline{\mu}(\widetilde{J})\right],$$

where

$$\widetilde{J}(s)=J(s)\log s,\ 0<s<1,$$

and

$$c_{i,n}=\int_{(i-1)/n\lambda_n}^{i/n\lambda_n}\widetilde{J}(s)ds,\ i=1,\ldots,n\lambda_n.$$

We can write then that for a large *n*

$$(n\lambda_n)^{1/2}\left[R(\lambda_n)\right]^{-2}\Delta_{n2}=2(1+o(1))\left[R(\lambda_n)\right]^{-1}$$
$$\times(n\lambda_n)^{1/2}\left[L_n(c)-\overline{\mu}(\widetilde{J})\right]$$

It's clear now, from corollary 2.3, that the right-hand-side of last expression converges in distribution to

$$2\int_0^1 s^{-1}\widetilde{J}(s)W(s)ds, \text{ as } n\to\infty,$$

which proves this lemma.

LEMMA 4.3.3. Let *(K)*, *(F)* and (H2) be satisfied. Then

$$(n\lambda_n)^{1/2}\left[R(\lambda_n)\right]^{-2}\Delta_{n3} = o_p(1) \quad \text{as} \quad n\to\infty.$$

Proof. It's straightforward, by applying still corollary 2.3. □

Recall that $K(0)=0$ and $K(1)=1$.

LEMMA 4.3.4. Let *(K)*, *(F)* and (H2) be satisfied. Then as $n\to\infty$

$$(n\lambda_n)^{1/2}\left[R(\lambda_n)\right]^{-2}\Delta_{n4} \xrightarrow{D} -2I(J)\int_0^1 s^{-1}J(s)W(s)ds,$$

$$I(J) = -\int_0^1 J(s)\log s ds.$$

Proof. It suffices to apply (1.6), (1.7) and corollary 2.3 and using the fact that

$$\int_0^1 s^{-1}K(s)ds = -\int_0^1 J(s)\log s ds.$$

This last yields by an integration by part. □

Finally, from lemmas 4.3.2 and 4.3.4, we write then that as $n\to\infty$,

$$(n\lambda_n)^{1/2}\left[R(\lambda_n)\right]^{-2}(D_n(b)-\Delta_{n5}) \xrightarrow{D} \int_0^1 s^{-1}\psi(s)W(s)ds,$$

$\psi(.)$ is as in theorem 3, which achieves proof of lemma 4.3.3 and consequently the proof of theorem 3. □

5. PROOFS\ OF COROLLARIES

5.1 PROOFS OF COROLLARIES 2.1 AND 2.2.

The proofs of corollaries 2.1 and 2.2 are immediate from the definition of the Wiener process. In fact we have

$$Cov(W(s),W(t)) = \min(s,t), \quad \text{for } 0<s<1, \;\; 0<t<1.$$

Let $\Xi = \int_0^1 s^{-1}J(s)W(s)ds$. Then we have

$$Var(\Xi) = E(\Xi)^2 = E\left(\int_0^1 s^{-1}J(s)W(s)ds\right)^2$$

$$= \int_0^1\int_0^1 s^{-1}t^{-1}J(s)J(t)Cov(W(s),W(t))dsdt$$
$$= \int_0^1\int_0^1 s^{-1}t^{-1}J(s)J(t)\min(s,t)dsdt$$
$$=: \sigma^2(J).$$

Since $(W(s),\ 0<s<1)$ is $N(0,1)$, then Ξ is also $N(0,\sigma^2(J))$ which achieves proof of corollary 2.1, and consequently by a same arguments the proof of corollary 2.2. □

($Cov(Y_1,Y_2)$ (resp. $Var(Y_1)$) denote the covariance of the couple of random variables (Y_1,Y_2) (resp. the variance of the random variable (Y_1)).

5.2 PROOFS OF COROLLARIES 2.3 AND 2.4

They are straightforward, it suffices to take in theorem 1 and 2 the weighting function $k_n\int_{(i-1)/k_n}^{i/k_n} J(s)ds$ instead of $J(i/k_n)$, and follow the same representation technics as in theorem 3. This completes the proofs of corollaries 2.3 and 2.4. □

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