PRESENTATION OF TOW SABIR ELEMENT RESULTS WITH IRREGULAR SHAPES

عرض لتأثير الشكل الهندسي الكيفي على عنصري **SABIR** المحدودين

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ABSTRACT

In 1985, Sabir developed two membrane finite elements having an additional nodal degree of freedom (DRILLING ROTATION). The main objective of this important development is to contribute in modeling the complex structures having only simple geometrical shape. In this paper, a new analytical integration expression is developed in order to model structures have complex geometrical shape. It is of importance to know how these elements will behave when they have irregular shapes.

ملخص

في سنة 1985 طـور الباحث **SABIR** عنصرين غشائيين يحتوي كل منهما على درجة حرية إضافية (**DRILLINGROTATION**)، هـذا التطوير المهم كان الهدف منه المساهمة في تمثيل المنشآت المركبة ذات الشكل الهندسي البسيط فقط. لتفادي هذا النقص من الناحية الهندسية نقترح في هذا البحث استعمال علاقة تكامل جديدة تسمح للعنصرين المذكورين بأخذ أشكال كيفية تساعد بصورة فعالة في تمثيل المنشآت المعقدة هندسيا. ترى كيف سيكون سلوك هذين العنصرين في هذه الحالة.

KEY WORDS

Strain Model, Analytical integration, Rectangular element, Triangular element, Drilling rotation, Irregular shapes.

1 INTRODUCTION:

To model a structure, which has complex geometrical shape in real problem, by a limited number of elements, already formulated to be applied as simple triangular or rectangular shape is not sufficient at all; furthermore, imagine how they can be used for complex structures.

Investigations at Cardiff University on the suitability of the available finite elements for curved structures, showed that to obtain satisfactory converged results, the assumed displacement elements required the curved structure to be divided into a large number of elements. Consequently, the strain-based approach was developed, not only for curved but also for flat elements. The approach is based on calculating the exact terms representing all the rigid body modes and the other components of the displacement functions are based on assumed independent strain functions insofar as it is allowed by the elasticity compatibility equations. This approach usually leads to the representation of the displacements by a higher order polynomial terms without the need for the introduction of additional internal and unnecessary degrees of freedom. Also faster convergence is usually obtained when the results are compared with the corresponding displacement i.e displacement elements having the same total number of degrees of freedom.

In the present paper, a triangular and rectangular element having the in-plane rotation as a degree of freedom are developed [1] using the strain model to extend their applications domain for the curved structures ; ie triangular or quadrilateral element whatever the geometrical shape of the element.

Hence, for reasons of importance and particularity of these elements (contain higher order shape functions expressed in terms of independent strains); it is indispensable to introduce irregular forms, which require a special integration technique, also a specific classification in programming level for different geometric forms is needed. The performance of these elements, using the new integration technique, is tested by applying them to the analysis of the problems used in previous publications and to obtain solutions for practical problems in engineering.

2 NUMERICAL INTEGRATION

The element stiffness matrix can be calculated using the following Eq.(1)

$$[K_e] = \left[A^{-1}\right]^T \iiint [B]^T [D][B] dv [A]^{-1} \tag{1}$$

To carry out the integral, we have to choose either numerical integration (e.g Gauss integration) or analytical integration. One of the disadvantage of the numerical integration is the high order of the monomials after the three multiplications of integral matrices Eq.(1), which would signify many integration points.

3 NEW APPROACH [2]

The numerical integration is usually the most frequent method used for displacement model to evaluate a polynomial of order (2n-1); with n integration points. On the other hand, if strain model is used where a high–order integration is employed, and displacement functions are coupled, the passage to the natural system coordinates will not be easy. The integration will be done analytically for a regular form, and the element stiffness terms are given implicitly in [2].

This new analytical integration expression allows **SBRIER** and **SBTIER** elements to have any distorted shape, and we will baptize them **SBQIER** and **SBTIER*** element**.**

With the following procedure, we have:

[Ke]=[A-1]T[K0][A-1] (2)

$[k_0]$ is given by the following expression:

$$[K_0] = t \iint_s [Q]^T [E][Q] dxdy \tag{3}$$

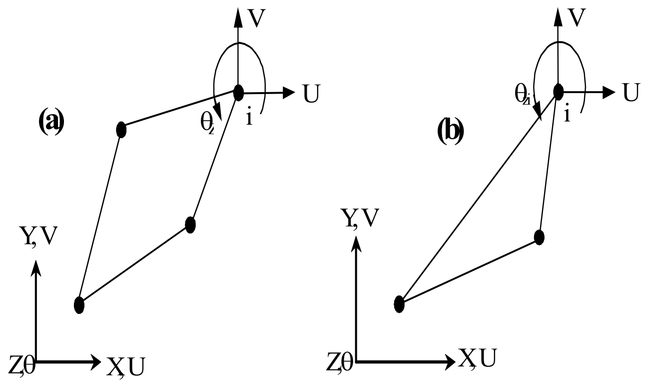

Fig 1 : *Quadrilateral and triangular elements*

Let I be the integral of the monomial c $x^\alpha y^\beta$ over the element surface.

$$I = c \iint_s x^\alpha y^\beta dxdy \qquad \text{c =constant}$$

The evaluation of the expression Eq.(3) always refers to the calculating of the integral I.

As it is known that the triangle is a particular case of the quadrilateral, therefore we will present the procedure used for the last one only.

The new integration technique is based on the three parts of **I** [2 , 3]**.**

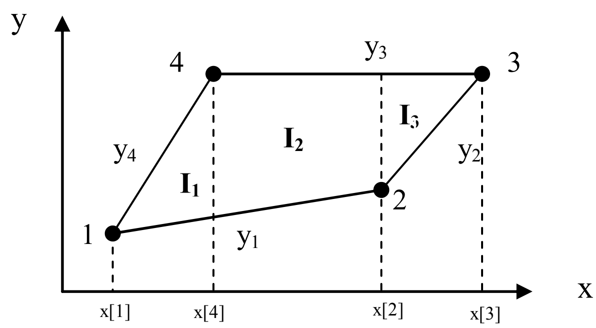

Fig 2 : Quadrilateral element

Therefore $I = I_1 + I_2 + I_3$

Refers to Fig.2:

$$I_1 = c \int_{x_1}^{x_4} \int_{y_1}^{y_4} x^\alpha y^\beta dxdy$$

$$I_2 = c \int_{x_4}^{x_2} \int_{y_1}^{y_3} x^\alpha y^\beta dxdy$$

$$I_3 = c \int_{x_2}^{x_3} \int_{y_2}^{y_3} x^\alpha y^\beta dxdy$$

With $y(x) = a\,x + b$

In general form :

$$y^{\beta} = \sum_{k=1}^{\beta+1} C(k)\cdot a^{\beta+1-k}\cdot b^{k-1}\cdot x^{\beta+1-k}$$

$$= \sum_{k=1}^{\beta+1} C(k)\cdot a^{k-1}\cdot b^{\beta+1-k}\cdot x^{k-1} \qquad (4)$$

C(k): coefficient in function of β [2)].

With : $\int y^{\beta}dy = \dfrac{1}{\beta+1}y^{\beta+1}$

$$= \frac{1}{\beta+1}\sum_{k=1}^{\beta+2} C(k)\cdot a^{k-1}b^{\beta+2-k}x^{k-1}{}^{\beta+1}$$

Therefore:

$$\int_{y_i}^{y_j} y^{\beta}dy = \frac{1}{\beta+1}\sum_{k=1}^{\beta+2} C(k)(a_j^{k-1}b_j^{\beta+2-k} - a_i^{k-1}b_i^{\beta+2-k})x^{k-1} \quad (5)$$

$$\iint x^{\alpha}y^{\beta}dxdy = \frac{1}{\beta+1}\sum_{k=1}^{\beta+2}\frac{1}{k+\alpha}C(k)(a_j^{k-1}b_j^{\beta+2-k} - a_i^{k-1}b_i^{\beta+2-k})(x_n^{k+\alpha} - x_m^{k+\alpha})$$

The general expression of I_p is :

$$I_p = \frac{C}{\beta+1}\sum_{k=1}^{\beta+2}\frac{1}{k+\alpha}C(k)(a_j^{k-1}b_j^{\beta+2-k} - a_i^{k-1}b_i^{\beta+2-k})(x_n^{k+\alpha} - x_m^{k+\alpha})...(6)$$

Hence $\qquad I = \sum_{p=1}^{3} I_p$

The integral limits are dependent on the specific form, the following **Figs. (3,4,…10)** show all the possibilities which could be met for quadrilateral shapes.

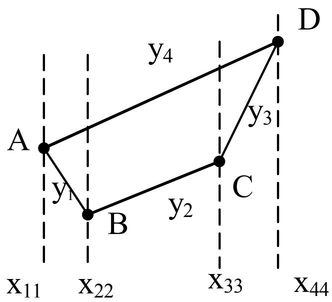

Fig.3 : *Shape 1*

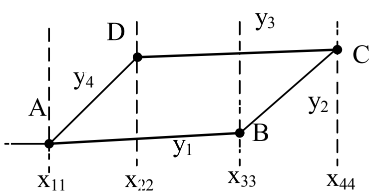

Fig.4 : *Shape 2*

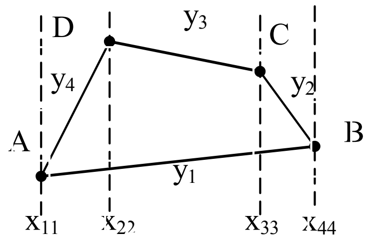

Fig.5 : *Shape 3*

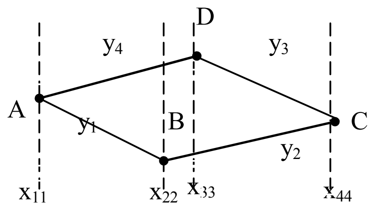

Fig.6 : *Shape 4*

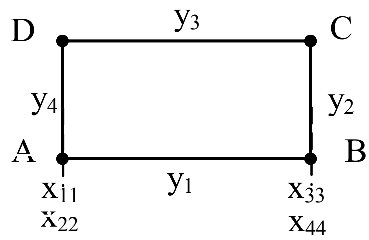

Fig.7 : *Shape 5*

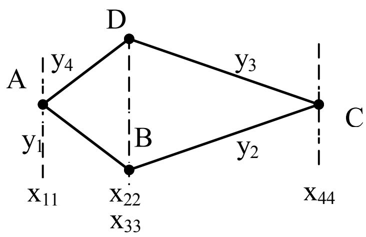

Fig.8 : *Shape 6*

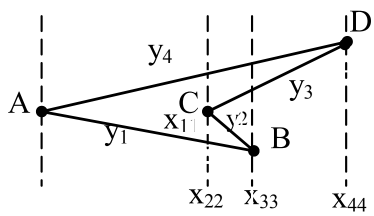

Fig.9 : *Shape 7*

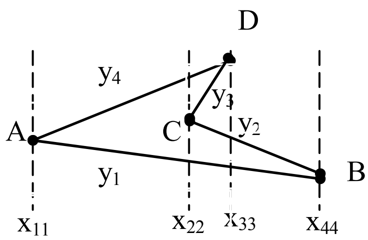

Fig.10 : *Shape 8*

For the case of triangle shapes, we have the following Figs. (11,12).

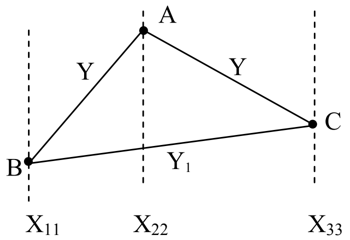

Fig.11 : Shape 1

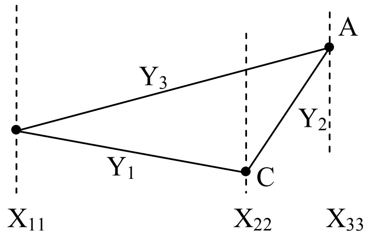

Fig.12 : *Shape 2*

4 SABIR ELEMENTS

4.1 Rectangular element (SBRIER) [4]

In this section we discus the SBRIER (Strain Based Rectangular In-plane Element Rotation) element formulated by Sabir**,** it is a rectangular element with four corner nodes and two degrees of freedom at each node, due to its limited performance which appears from its restricted geometric, the element is strictly applied in rectangular structures or sectors (with the 2nd formulation) [5], more than that this element can not be used for curved plate structures.

The following expressions for strains are proposed by Sabir**.**

$$\varepsilon_x = a_4 + a_5y + (a_{11}y^2 + 2a_{12}x\,y^3)$$
$$\varepsilon_y = a_6 + a_7x + (-a_{11}x^2 - 2a_{12}x^3y) \qquad (7)$$
$$\gamma_{xy} = a_8 + a_9x + a_{10}y + (a_5x + a_7y)$$

The final shape function for the rectangular element will be given by:

$$u = a_1 - a_3y + a_4x + a_8y/2 + a_5xy + a_{10}y^2/2 + a_{11}xy^2 + a_{12}x^2y^3$$
$$v = a_2 + a_3x + a_6y + a_8x/2 + a_7xy + a_9x^2/2 - a_{11}x^2y - a_{12}x^3y^2 \qquad (8)$$
$$\phi = a_3 - a_5x/2 + a_7y/2 + a_9x/2 + a_{10}y - 2a_{11}xy - 3a_{12}x^2y^2$$

4.2 Triangular element (SBTIEIR) [4]

The following expressions for strains are proposed by Sabir**.**

$$\varepsilon_x = a_4 + a_5y + a_7\,x$$
$$\varepsilon_y = a_6 + a_7x + a_5y \qquad (9)$$
$$\gamma_{xy} = a_8 + a_9\,(x+y)$$

The final shape function for the triangular element will be given by the following equations :

$$u = a_1 - a_3\,y + a_4\,x + a_8y/2 + a_5\,xy + a_7(x^2 - y^2)/2 + a_9\,y^2/2$$
$$v = a_2 + a_3\,x + a_6\,y + a_8x/2 + a_5(y^2 - x^2)/2 + a_7xy + a_9x^2/2 \qquad (10)$$
$$\phi = a_3 - a_5\,x + a_7\,y + a_9\,(x - y)/2$$

We should notice that displacement functions contain quadratic terms to allow for change in curvature.

The element stiffness matrix $[K_e]$ can be calculated following the usual finite element technique and using the following equations.

$$[K_e] = [A^{-1}]^T [K_0] [A^{-1}] \qquad (11)$$

With

$$[K0] = \iint_S [Q]^T [D].[Q].\mathrm{d}\,x.\mathrm{d}\,y \qquad (12)$$

To allow the elements have a quadrilateral and any triangular shape, we will use the new approach in the following tests.

5 APPLICATIONS

5.1 High order Patch Test : Pure bending of a cantilever beam.

This example was also treated by Ibrahimbégivic, Frey and Rebora in their recent paper of synthesis [6], in order to show the performance of the finite elements with traditional formulation but with not-conventional interpolations. They took ν=0 (although it is about a real plane problem).

A cantilever beam with rectangular section (L x T x H = 10 x 1 x 2) is subjected to a pure bending formed by two nodal forces (P=10) forming a couple (consisting loading case).

The loading case cc1 Fig.13 represents the high order Patch –Test [7], two rectangular and four triangular elements of membrane (regular mesh) model the cantilever beam.

Let us compare the behavior of SBQIEIR and SBTIEIR* with the robust element of Ibrahimbégovic et al. [6].

According to this load pattern, it can be says that the cantilever beam is subjected to pure bending and it can be translates this fact by supposing that the only no null stress is σ_{xx}.

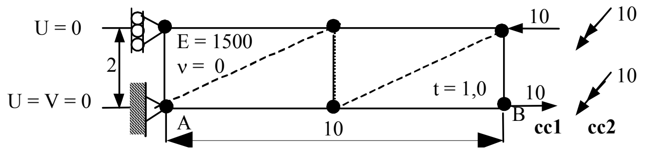

***Fig. 13 :** Pure bending of a cantilever beam. Data (cc = load case)*

Table.1 gives the results obtained with the regular mesh. For SBQIEIR element, the consisting loading case (cc1) provides the exact solution; the error is very small in the case of inconsistent load cc2 (but.quite practical!). On the other hand in the case of the SBTIEIR* element the results obtained are very far from the analytical solution for the two cases of loading (regular mesh!).

***Table 1 :** Pure bending of a cantilever beam; regular mesh.*

	SBQIEIR			SBTIEIR*			Ibrahimbegovic et al. Réf. [6]		
Loading case	V_B	θ_{zB}	σ_{Xa}	V_B	θ_{Zb}	σ_{xA}	V_B	ψ_B	σ_{xA}
1 : Couple	1,0	0,2	30,0	0.58	0,12	21.04	1,0	0,2	30,0
2 : Moment	1,0067	0,202	30,0	0,58	0,12	21,11	1,0067	0,2017	30,0
Beams theory Réf. [6]	1,0	0,2	30,0	1,0	0,2	30,0	1,0	0,2	30,0

5.2 MacNeal's elongated beam

MacNeal and Harder [8] cantilever beam of dimension (6 x 2 x 1) whose details are given in Fig.14 is subjected to end bending moment (M=10), and applied charge at free end (P=1).

The cantilever is modeled by six rectangular (Fig.14a), trapezoidal (Fig.14b) and parallelogram (Fig.14c) membrane elements.

Tableau 2 : Tip deflection for MacNeal's elongated beam.

Element Model	Pure bending			End shear		
	Regular	Trapezoidal	Parallel	Regular	Trapezoidal	Parallel
Q4	0,0251	0,0059	0,0084	0,0100	0,0029	0,0037
PS5β	0,270	0,0124	0,1960	0,1073	0,0056	0,0683
AQ	0,2457	0,2206	0,2379	0,0977	0,0871	0,0944
MAQ	0,2457	0,2393	0,2403	0,0977	0,0943	0,0955
Q4S	-	-	-	0,1073	0,1066	0,1068
07β	0,270	0,2694	0,2678	0,1073	0,1068	0,1065
SBQIEIR	0,2670	0,2667	0,2667	0,1042	0,1027	0,1027
SBTIEIR*	0.1180	0.0038	0.1010	0.0469	0.0005	0.0357
Beams Theory	**0,270**			**0,1081**		

MacNeal [9] has proved that analysis using the trapezoidal mesh always leads to the kind of unsatisfactory performance termed trapezoidal locking.

The results obtained for **SBQIEIR and SBTIEIR*** elements are compared with the others obtained from the known quadrilateral finite elements Table 2.

From these three mesh cases (Fig14.a, 14b, and 14c), we proved the efficiency of the present element **SBQIEIR.** From the beam tip deflections in Table 2, it can be seen that all elements with drilling degrees of freedom can circumvent the trapezoidal locking whilst Q4 and PS5β will always lock [9].

We come to the conclusion that **SBQIEIR** element presents more performance than other elements for this type of bending problems, and still stable despite geometric distortion, whilst a bad results are obtained with **SBTIEIR*** element.

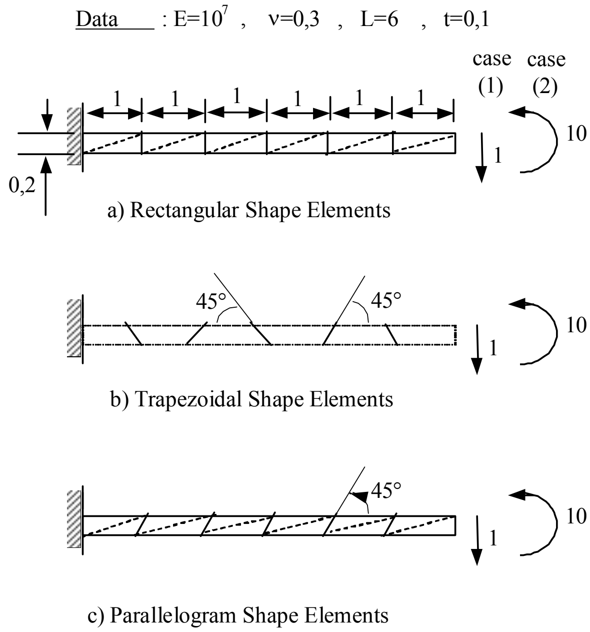

***Fig.14 :** MacNeal's elongated beam. Data and mesh.*

5.3 Allman's cantilever beam

It is useful to know how behaves a finite element presents a significant geometrical distortion. Sze, Chen and Cheung studied this problem [10], in order to test the performance and the robustness of elements 07β et 07β*

It is a question of evaluating vertical displacement V_A at the free end of a cantilever beam (Fig.15) subjected to a uniform vertical load (of resultant W).

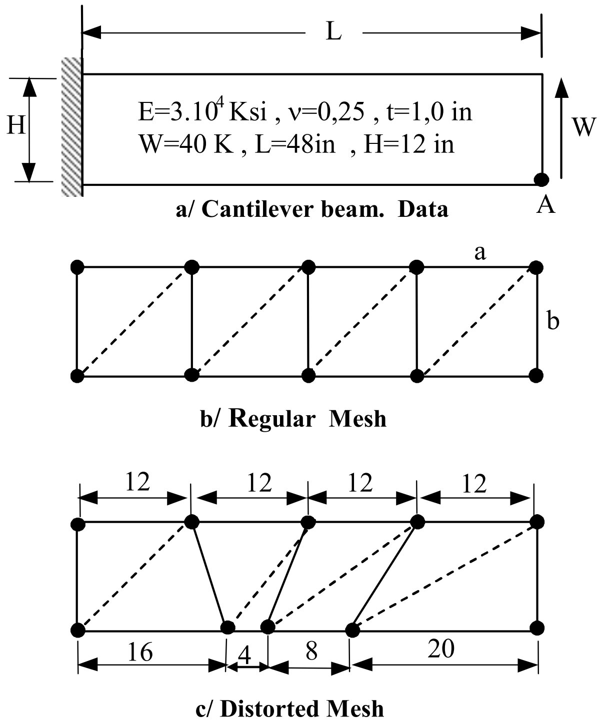

Fig.15 : Allman's cantilever beam; Data and mesh.

The researchers use this example as a test to validate the plane elements. It possible to examine the aptitude of an element of the membrane type to simulate problems dominated by the bending. These elements **SBQIEIR** and **SBTIEIR*** are compared with the analytical solution (exact) given by Timoshenko and Goodier [11] and with other known elements :

$$V_A = \frac{PL^3}{3EI} + \frac{(4+5\upsilon)}{2EH} PL = 0{,}3553 \qquad (13)$$

The results obtained for the two cases of mesh (regular and distorted) are presented in Table 3.

In the case of the regular mesh Fig15b, the results obtained for **SBQIEIR** are powerful and comparable with the analytical solution given by the beams theory contrary to **SBTIEIR*** element. For the case of the distorted mesh Fig14c the very good performance of **SBQIEIR** element is confirmed. The corresponding results are more precise than the results given by **SBTIEIR *** element (see table 3).

Table.3 : Allman's cantilever beam. Vertical displacement at A.

Element /Formulation	Mesh	Vertical displacement at A
Q4	Reg.	0,2412
Q4	Dist.	0,2117
PS5β	Reg.	0,3475
PS5β	Dist.	0,3286
AQ	Reg.	0,3261
AQ	Dist.	0,3365
MAQ	Reg.	0,3262
MAQ	Dist.	0,3382
QR4b	Reg.	0,3475
QR4b	Dist.	0,3471
Q4S	Reg.	0,3475
Q4S	Dist.	0,3467
07β	Reg.	0,3475
07β	Dist.	0,3475
Ref. [12]	Reg.	0,3443
Ref. [12]	Dist.	0,3066
Ref. [8]	Reg.	0,3407
Ref. [8]	Dist.	0,2977
Ref. [13]	Reg.	0,3027
Ref. [13]	Dist.	-
Ref. [14]	Reg.	0,3507
SBQIEIR	Dist.	0,3482
Ref. [14]	Reg.	0.1389
SBTIEIR*	Dist.	0.1400
Exact solution according Ref [11] :		0,3553

6 CONCLUSION

- It can be said that strain model has been dominated nowadays, because it allows the displacement to be represented by a higher order polynomial terms without the need for the introduction of the degrees of freedom or additional nodes.

- The robustness of the present element **SBQIER** via the distorted mesh was shown, this is due probably to the analytical technique of integration used contrary to triangular element **SBTIER*** remains very sensitive to the mesh distortion, the results are still bad even for a regular mesh; thus, practically it is advised to use **SBQIER** element.

REFERENCES

[1] Sabir, A.B. (1984) strain based finite elements for the analysis of shear walls, *Proc.Int.Conf. on Tall buildings*, 447-453.Hong Kong University.

[2] Belarbi, M.T. (2000) *Développement de nouveaux éléments finis à modèle en déformation, application linéaire et non linéaire*. Thèse de Doctorat, Université de Mentouri Constantine.

[3] Sedira, L. (2002) *Contribution du modèle en déformation dans l'analyse des plaques stratifiées composites, thèse de Magister*, Université Colonel El Hadj Lakhdar Batna, Algerie.

[4] Sabir, A.B. (1985) A rectangular and triangular plane elasticity element with drilling degrees of freedom, *Chapter 9 in proceeding of the 2nd international conference on variational methods in engineering, Southampton* University, Springer-Verlag, Berlin, pp. 17-25.

[5] Sabir, A.B. et Salhi, H.Y. (1986), A strain based finite element for general plane elasticity in polar coordinates, *Res. Mechanica* 19, pp. 1-16.

[6] Ibrahimbegovic A., Frey F. et Rebora B., Une approche unifiée de la modélisation des structures complexes : les éléments finis avec degré de liberté de rotation, LSC Rapport Interne 93/10, Ecole polytechnique fédérale de Lausanne (Suisse), Juin 1993

[7] Taylor R.L., Simo J.C., Zienkiewicz O.C. and Chan A.C., The patch test: A Condition for Assessing Finite Element Convergence, IJNME, Vol. 22, pp. 39-62, 1986.

[8] MacNeal R. H. et Harder R. L., A refined four-noded membrane element with rotational degrees of freedom, C.S., Vol. 28, pp. 75-84, 1988.

[9] MacNeal R. H., A theorem regarding the locking of tapered four-noded membrane elements, IJNME., Vol. 24, pp. 1793-1799, 1987

[10] Sze K.Y., Chen W. and Cheung Y.K., An efficient quadrilateral plane element with drilling degrees of freedom using orthogonal stress modes, C.S., Vol. 42, N° 5, pp. 695-705, 1992.

[11] Timoshenko S. and Goodier J. N., Theory of Elasticity, Mc Graw-Hill, New York, 1951.

[12] Ibrahimbegovic A, Taylor R.L. et Wilson E.L., A robust quadrilateral membrane finite element with drilling dégréés of freedom, IJNME, Vol. 30, pp.445-457, 1990.

[13] Allman D.J., A quadrilateral finite element including vertex rotations for plane elasticity analysis, IJNME, Vol. 26, pp. 717-730, 1988.

[14] Sabir A.B., A rectangular and triangular plane elasticity element with drilling degrees of freedom, Chapter 9 in proceeding of the 2nd international conference on variational methods in engineering, Southampton University, Springer-Verlag, Berlin, pp. 17-25, 1985.