IJAMS
Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

Stochastic Calculus with a Special Generalized Fractional Brownian Motion

Mounir Zili[1]

ABSTRACT: This work is a first step toward developing a stochastic calculus theory with respect to the generalized fractional Brownian motion, which is a recently introduced Gaussian process extending both fractional and sub-fractional Brownian motions. A Malliavin divergence operator and a Stochastic symmetric integral with respect to this process are defined, and sufficient integrability conditions are provided. Moreover, corresponding Itô formulas are established, then applied to introduce a generalized version of the fractional Black–Scholes option pricing model.

Keywords: Fractional, Sub-frational, Brownian motion, Malliavin Calculus, Stochastic Symmetric integral, Black-Scholes equation.

MSC: 60G15, 60G22, 60H05

1 Introduction

The theory of Stochastic integration with respect to Gaussian processes has been recently reinforced by different approaches, with a particular attention on the cases of the fractional and sub-fractional Brownian motions due to the significant applications of these processes in practical phenomena such as telecommunications, hydrology or economics. Some surveys and the complete literature could be found e.g. in [1], [2], [7], [10], [16], [17].

This paper is devoted to lay the first milestones of a stochastic calculus with respect to the generalized fractional Brownian motion introduced by M. Zili in [18], which is a Gaussian process extending both fractional and sub-fractional Brownian motions. First, let us recall that the two-sided fractional Brownian motion (tsfBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$, defined on a probability space $(\Omega, F, \mathbb{P})$, with the covariance function:

$$Cov(B_t^H, B_s^H) = \frac{1}{2}\big(\mid s \mid^{2H} + \mid t \mid^{2H} - \mid t-s \mid^{2H} \big);\ t,s \in \mathbb{R} \tag{1.1}$$

The restriction of the tsfBm to the set $[0,+\infty)$ is the well known fractional Brownian motion (fBm), which in turn is an extension of the Brownian motion (Bm) because $Cov(B_s^{1/2}, B_t^{1/2}) = t \wedge s$ for every $s,t \geq 0$. Both tsfBm and fBm have been considered as an important tool in modeling due to their properties of long-range dependence, self-similarity and stationarity of their increments. For more information on tsfBm and fBm see, e.g. [6, and references therein].

The sub-fractional Brownian motion (sfBm) was introduced in [3], as an extension of the Browian motion, preserving most of the properties of the fBm, but not the stationarity of the increments. It is the stochastic process $\xi^H = \{\xi_t^H, t \geq 0\}$, defined by:

$$\forall t \in \mathbb{R}_+, \qquad \xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}},$$

• [1] *Mounir Zili, corresponding author, University of Monastir, Faculty of sciences of Monastir, Laboratory LR18ES17, Avenue de l'environnement, 5000 Monastir, Tunisia.*
E-mail: Mounir.Zili@fsm.rnu.tn

Communicated Editor: Mezerdi Brahim
Manuscript received Dec 03, 2023; revised Feb 02, 2024; accepted Feb 02, 2024; published May 05, 2024.

where B^H is the tsfBm defined above. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. We refer to [3], [15, and references therein] for further information on the sfBm.

In the last decades, many extension kinds of fBm and sfBm processes have been introduced such as the multifractional Brownian motion [11], the mixed fractional Brownian motion [20], the bifractional Brownian motion [5], the mixed sub-fractional Brownian motion [4], [8], [21] and the generalized sub-fractional Brownian motion [13]. But all these extensions do not generalize fBm and sfBm in the same time: each of them extends either only fBm or only sfBm.

In [18], M. Zili has introduced a new extension of both fBm and sfBm processes, completely different from all the above cited processes. The Zili's generalized fractional Brownian motion (ZgfBm) of parameters a, b and H is the process $Z^H(a,b) = \{Z_t^H(a,b); t \geq 0\}$, defined on the probability space $(\Omega, F, \mathbb{P})$ by:

$$\forall t \in \mathbb{R}_+, \quad Z_t^H(a,b) = aB_t^H + bB_{-t}^H, \tag{1.2}$$

where $(B_t^H)_{t\in\mathbb{R}}$ is a tsfBm of parameter H. If $a = 1$ and $b = 0$, $Z^H(a,b)$ is clearly a fBm, and if $a = b = \frac{1}{\sqrt{2}}$, $Z^H(a,b)$ is a sfBm. Therefore, the ZgfBm is in the same time, an extenstion of the fBm and of the sfBm, and this can be considered as a main mathematical motivation for the study of such process. In addition, the ZgfBm should allow researchers to deal with a larger class of modeled natural phenomena, including those with stationary or with non-stationary increments, whereas with fBm we can model only phenomena with stationary increments, and sfBm allows us to model only phenomena with non-stationary increments. This is a second main motivation for constructing a stochastic calculus with respect to the ZgfBm.

Moreover, the dependence of $Z^H(a,b)$ on three parameters a, b and H, should allow us to construct more adequate models, allowing for example to control the level of correlation between the increments of the studied phenomena. This feature is not available with already known fractional and sub-fractional Brownian motions, and consequently it should overcome the deficiency of fBm and sfBm models due to their dependence on one single constant H. More information about ZgfBm can be found in [18], [19]

In this paper, our purpose is to lay the first milestones of a stochastic analysis of this process. We first prove the existence of a Malliavin divergence operator and a stochastic symmetric integral with respect to ZgfBm when $H > 1/2$. The first approach of integration has been largely used in literature [2], [10] and it is essentially based on the Malliavin calculus. While, the symmetric integral approach is due to Russo and Vallois [12], and it allows us, among other things, to interpret the Malliavin divergence operator with respect to ZgfBm as stochastic integral.

We should note that in [1], [16], a stochastic calculus was presented in the particular cases of fBm and sfBm, basing on the fact that B^H and ξ^H are Gaussian processes that can be written in the form

$$\int_0^t K(s,t)dW_s, \tag{1.3}$$

where W is a Wiener process and $K(s,t)$ is a square integrable kernel. In the first part of this paper, we follow a different approach, essentially inspired from [2], with which we introduce a stochastic integration with respect to the ZgfBm, without having to write the process $Z^H(a,b)$ in the form (1.3).

Then, in a second part, we derive an Itô-formula allowing us to get an expression of $f(t, Z_t^H(a,b))$ in function of the Malliavin divergence operator integral with respect to $Z^H(a,b)$, when $H > \frac{1}{2}$ and $f : (t,x) \longmapsto f(t,x)$ is a two-variables map statisfying some suitable smootheness assumptions. In the particular case where $f : x \longmapsto f(x)$ is a one-variable function, we get two Itô-formulas expressing $f(Z_t^H(a,b))$ in function of the Malliavin divergence and of the Stochastic symmetric integrals respectively. Our established formulas extend the results obtained in [2], [17]. We note that in contrast to many previous works, here we give a detailed proof for the more general Itô formula with $f(t, Z_t^H(a,b))$, then we deduce the particular form of $f(Z_t^H(a,b))$. In the other papers, generally the proofs are established only with $f(B_t^H)$ or $f(\xi_t^H)$, then the general formulas with $f(t, B_t^H)$ or $f(t, \xi_t^H)$ are simply stated. We also note that our proofs take the same strategies as in [2], but to overcome the difficulties due to the generalization procedure and especially to the dependence of $Z^H(a,b)$ of three parameters, we establish here some other technical tools.

The paper is organized as follows. In section 2 we recall some interesting stochastic characteristics of the ZgfBm, and we establish some useful new properties of this process. In section 3, we introduce a Malliavin divergence operator and a stochastic symmetric integral with respect to $Z^H(a,b)$, we justify the existence of both integrals and we establish an interesting relationship between them. Section 4 is devoted to establish Itô formulas with respect to ZgfBm, and to present some application examples. In particular, we introduce a new stochastic differential equation, generalizing the known fractional Black–Scholes option pricing model.

2 Useful ZgfBm's characteristics

We will first recall some interesting properties of the ZgfBm, all of them are obtained in [18].

Lemma 1. *The ZgfBm $(Z_t^H(a,b))_{t\in\mathbb{R}_+}$ is a centered Gaussian process, with covariance function*

$$\begin{aligned} & R^{H,a,b}(t,s) := Cov\Big(Z_t^H(a,b), Z_s^H(a,b)\Big) \\ = \; & \frac{1}{2}(a+b)^2\left(s^{2H}+t^{2H}\right) - ab(t+s)^{2H} - \frac{a^2+b^2}{2}\mid t-s\mid^{2H} . \end{aligned} \tag{2.1}$$

Moreover,

1) *The ZgfBm is a self-similar process; that is the processes $\{Z_{ht}^H(a,b); t\geq 0\}$, and $\Big\{h^H Z_t^H(a,b); t\geq 0\Big\}$ have the same law.*
2) *There exist two positive constants, $\gamma(a,b,H)$ and $\nu(a,b,H)$ such that, for all $(s,t)\in\mathbb{R}_+^2$; $s\leq t$,*

$$\gamma(a,b,H)(t-s)^{2H} \leq E\Big(Z_t^H(a,b)-Z_s^H(a,b)\Big)^2 \leq \nu(a,b,H)(t-s)^{2H}. \tag{2.2}$$

3) *The ZgfBm Z^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

From (2.1), the following variance expression is deduced:

$$E\left(Z_t^H(a,b)^2\right) := C_H(a,b)t^{2H}; \qquad C_H(a,b) = a^2+b^2-(2^{2H}-2)ab. \tag{2.3}$$

We will now establish other new characteristics of the covariance function $R^{H,a,b}$ of ZgfBm, that will play a great role in the sequel of this paper.

Lemma 2. *For every $(a,b)\in\mathbb{R}^2$, we have*

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s\partial t} = H(2H-1)\left[(a^2+b^2)|t-s|^{2H-2} - 2ab(t+s)^{2H-2}\right]. \tag{2.4}$$

Moreover, when $\frac{1}{2}<H<1$, denoting $\alpha_H = H(2H-1)$, we have

$$C_1|t-s|^{2H-2} \leq \frac{\partial^2 R^{H,a,b}(s,t)}{\partial s\partial t} \leq C_2|t-s|^{2H-2}, \tag{2.5}$$

for every $s,t\in[0,T]$; $s\neq t$, with

$$C_1 = \min(\alpha_H(a^2+b^2), \alpha_H(a-b)^2) \quad and \quad C_2 = \alpha_H(|a|+|b|)^2.$$

Proof. The explicit expression of $\dfrac{\partial^2 R^{H,a,b}}{\partial s\partial t}$ can be easily obtained. We will just prove the two stated estimates (2.5).

Since $|t-s|\leq(t+s)$ and $x\longmapsto x^{2H-2}$ is decreasing, we have

$$(t+s)^{2H-2} \leq |t-s|^{2H-2},$$

and consequently

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} \leq \alpha_H (|a|+|b|)^2 |t-s|^{2H-2}.$$

For the lower bound, if $ab<0$ then,

$$\alpha_H \left[(a^2+b^2)|t-s|^{2H-2} - 2ab(t+s)^{2H-2}\right] \geq \alpha_H (a^2+b^2)|t-s|^{2H-2}.$$

And if $ab \geq 0$ then, we can write

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} = \alpha_H \left[(a-b)^2|t-s|^{2H-2} + 2ab\left[|t-s|^{2H-2} - (t+s)^{2H-2}\right]\right]$$

which clearly implies that

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} \geq \alpha_H (a-b)^2 |t-s|^{2H-2}.$$

□

The following characteristic of the second partial derivative of $R^{H,a,b}$ will also be useful for the sequel of this article.

Lemma 3. *When $\frac{1}{2} < H < 1$, there exists a positive constant C_3, such that*

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \left|\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial s \partial t}\right| d\sigma \leq C_3 |t-s|^{2H-2}, \tag{2.6}$$

for every $\epsilon > 0$ and $s,t \in [0,T]$; $s \neq t$.

Proof. Using Expression (2.4) and the mean value theorem we get

$$\begin{aligned} & \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \left|\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial s \partial t}\right| d\sigma \\ = & \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \alpha_H \left[(a^2+b^2)|t-s-\sigma|^{2H-2} - 2ab(t+s+\sigma)^{2H-2}\right] d\sigma \\ = & \alpha_H \left[(a^2+b^2)|t-s-\sigma_\epsilon|^{2H-2} - 2ab(t+s+\sigma_\epsilon)^{2H-2}\right] \end{aligned} \tag{2.7}$$

with $\sigma_\epsilon \in (-\epsilon,\epsilon)$. Without loss of any generality, we assume that $t > s$, and we distinguish two cases:

First case: If $|t-s| \geq 2\epsilon$, then, by Lemma 2, we have

$$\alpha_H \left[(a^2+b^2)|t-s-\sigma_\epsilon|^{2H-2} - 2ab(t+s+\sigma_\epsilon)^{2H-2}\right] \leq C_2 |t-s-\sigma_\epsilon|^{2H-2}.$$

Moreover, $t-s-\sigma_\epsilon \geq t-s-\epsilon \geq \frac{t-s}{2}$ and $2H-2<0$. So

$$|t-s-\sigma_\epsilon|^{2H-2} \leq 2^{2-2H}|t-s|^{2H-2}$$

and therefore,

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial s \partial t} d\sigma \leq 2^{2-2H}|t-s|^{2H-2}.$$

Second case: If $|t-s| < 2\epsilon$ then, again by Lemma 2, we get

$$\begin{aligned} & \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \alpha_H \left|\left[(a^2+b^2)|t-s-\sigma|^{2H-2} - 2ab(t+s+\sigma)^{2H-2}\right]\right| d\sigma \\ & \leq C_2 \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} |t-s-\sigma|^{2H-2} d\sigma = C_2 I_\epsilon. \end{aligned}$$

We have $I_\epsilon = I_{\epsilon,1} + I_{\epsilon,2}$ with

$$I_{\epsilon,1} = \frac{1}{2\epsilon}\int_{-\epsilon}^{0} |t-s-\sigma|^{2H-2} d\sigma \ \ and \ \ I_{\epsilon,2} = \frac{1}{2\epsilon}\int_{0}^{+\epsilon} |t-s-\sigma|^{2H-2} d\sigma.$$

When $-\epsilon < \sigma < 0$, we have $t-s-\sigma > t-s$, which yields

$$|t-s-\sigma|^{2H-2} \leq |t-s|^{2H-2}$$

and, therefore $I_{\epsilon,1} \leq \frac{1}{2}|t-s|^{2H-2}$.

Now to bound $I_{\epsilon,2}$, we will distinguish two sub-cases:

If $\epsilon \leq |t-s| < 2\epsilon$, then

$$\begin{aligned} I_{\epsilon,2} &= \frac{1}{2\epsilon}\int_0^{\epsilon}(\sigma-(t-s))^{2H-2}d\sigma \\ &= \frac{1}{2\epsilon(2H-1)}\left((t-s)^{2H-1}-((t-s)-\epsilon)^{2H-1}\right) \\ &\leq \frac{1}{2\epsilon(2H-1)}(t-s)^{2H-1} \\ &\leq \frac{1}{2H-1}2^{2H-2}\epsilon^{2H-2} \\ &\leq \frac{1}{2H-1}|t-s|^{2H-2}, \end{aligned}$$

where in the second inequality we used the fact that $\epsilon \leq |t-s|$, and in the third one is due to the fact that $|t-s| < 2\epsilon$.

If $|t-s| < \epsilon$, then

$$\begin{aligned} I_{\epsilon,2} &= \frac{1}{2\epsilon}\int_0^{t-s}((t-s)-\sigma)^{2H-2}d\sigma + \frac{1}{2\epsilon}\int_{t-s}^{\epsilon}(\sigma-(t-s))^{2H-2}d\sigma \\ &= \frac{1}{2\epsilon(2H-1)}\left((t-s)^{2H-1}+(\epsilon-(t-s))^{2H-1}\right) \\ &\leq \frac{1}{2\epsilon(2H-1)}\left((t-s)^{2H-1}+\epsilon^{2H-1}\right) \\ &\leq \frac{1}{2H-1}\epsilon^{2H-2} \\ &\leq \frac{1}{2H-1}|t-s|^{2H-2}, \end{aligned}$$

where the two last inequalities are due to the fact that $|t-s| < \epsilon$.

Therefore, $I_\epsilon \leq (\frac{1}{2}+\frac{1}{2H-1})|t-s|^{2H-2}$.

All this yields Inequality (2.6) with

$$C_3 = \max\left(\frac{1}{2}+\frac{1}{2H-1}\ ,\ 2^{2-2H}\right).$$

□

We will finish this section by the following useful lemma.

Lemma 4. *For every integer $p > 0$ and real $\lambda > 0$ satisfying $p\lambda < \frac{T^{-2H}}{2C_H(a,b)}$ we have*

$$\mathbb{E}\left(\exp\left(p\lambda \sup_{0\leq t\leq T}|Z_t^H(a,b)|^2\right)\right) < \infty.$$

Proof. Since $Z^H(a,b)$ is a centered Gaussian process, from Theorem 4.2 in [9], we know that $m := \mathbb{E}\left(\sup_{u\in[0,1]} Z_u^H(a,b)\right)$ is finite and, for all $x > m$ we have

$$\mathbb{P}\left(\sup_{u\in[0,1]} Z_u^H(a,b) \geq x\right) \leq \exp\left(-\frac{(x-m)^2}{2\sup_{u\in[0,1]} Var(Z_u^H(a,b))}\right). \tag{2.8}$$

Using the fact that $Z^H(a,b)$ and $-Z^H(a,b)$ have the same law, Equation (2.8) and Expression (2.3), we get that, for any $x > m^2$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{u\in[0,1]} \left|Z_u^H(a,b)\right|^2 \geq x\right) &= \mathbb{P}\left(\sup_{u\in[0,1]} \left|Z_u^H(a,b)\right| \geq \sqrt{x}\right) \\
&\leq 2\mathbb{P}\left(\sup_{u\in[0,1]} Z_u^H(a,b) \geq \sqrt{x}\right) \\
&\leq 2\exp\left(-\frac{(\sqrt{x}-m)^2}{2C_H(a,b)}\right).
\end{aligned} \tag{2.9}$$

Now, by the self similarity property (cf. Assertion 1 in Lemma 1) and using Equation (2.9) we get

$$\begin{aligned}
\mathbb{E}\left(e^{p\lambda \sup_{0\leq t\leq T} |Z_t^H(a,b)|^2}\right) &= \mathbb{E}\left(e^{p\lambda T^{2H} \sup_{0\leq t\leq 1} |Z_t^H(a,b)|^2}\right) \\
= \ & \int_0^{+\infty} p\lambda T^{2H} \exp\left(p\lambda T^{2H} x\right) \mathbb{P}\Big(\sup_{u\in[0,1]} \left|Z_u^H(a,b)\right|^2 \geq x\Big) dx \\
\leq \ & 2p\lambda T^{2H} \int_{m^2}^{+\infty} \exp\left(\gamma_H x + \frac{m}{C_H(a,b)}\sqrt{x} - \frac{m^2}{2C_H(a,b)}\right) dx \\
& + p\lambda T^{2H} \int_0^{m^2} \exp\left(p\lambda T^{2H} x\right) dx
\end{aligned}$$

with $\gamma_H = p\lambda T^{2H} - \dfrac{1}{2C_H(a,b)}$. The functions $x \longmapsto \exp\left(p\lambda T^{2H} x\right)$ and $x \longmapsto \exp\left(\gamma_H x + \dfrac{m}{C_H(a,b)}\sqrt{x} - \dfrac{m^2}{2C_H(a,b)}\right)$ are continuous. Moreover,

$$\gamma_H x + \frac{m}{C_H(a,b)}\sqrt{x} - \frac{m^2}{2C_H(a,b)} \sim \gamma_H x \ \text{ as } x \to +\infty,$$

and $\gamma_H < 0$ because $p\lambda < \dfrac{T^{-2H}}{2C_H(a,b)}$. Therefre, the last two integrals are finite, and the proof of Lemma 4 is achieved. □

In the sequel of this paper, we fix $\frac{1}{2} < H < 1$.

3 Stochastic integration with respect to ZgfBm when $\frac{1}{2} < H < 1$.

We start this section by recalling some basic facts of the Malliavin calculus and by adapting them to the particular Gaussian process $Z^H(a,b)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where the $\sigma-$algebra $\mathcal{F}$ is generated by the tsfBm $(B_t^H)_{t\in\mathbb{R}}$ of parameter H. For every fixed $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we denote by $\mathcal{H}_{a,b}$ the canonical Hilbert space associated to the Gaussian process $Z^H(a,b)$. This Hilbert space is defined as the closure of the linear span generated by the indicator functions $\mathbf{1}_{[0,t]},\ t > 0$, with respect to the inner product:

$$< \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} >_{\mathcal{H}_{a,b}} = \mathcal{R}^{H,a,b}(t,s) = Cov(Z_t^H(a,b), Z_s^H(a,b)).$$

The isometry between $\mathcal{H}_{a,b}$ and Gaussian space associated with $Z^H(a,b)$ is denoted by $\varphi \longmapsto Z^H(a,b)(\varphi)$ which is extended by the mapping

$$\mathbf{1}_{[0,t]} \longmapsto Z_t^H(a,b).$$

For any $\varphi, \psi \in \mathcal{H}_{a,b}$,

$$< \varphi, \psi >_{\mathcal{H}_{a,b}} = \int_0^T \int_0^T \varphi(u)\psi(v) \frac{\partial^2 \mathcal{R}^{H,a,b}}{\partial u \partial v}(u,v) du dv.$$

Consider the subspace $|\mathcal{H}_{a,b}|$ of $\mathcal{H}_{a,b}$ which is defined as the set of measurables functions f on $[0,T]$ such that

$$\|f\|^2_{|\mathcal{H}_{a,b}|} = \int_0^T \int_0^T |f(u)|\ |f(v)|\ \left|\frac{\partial^2 R^{H,a,b}(u,v)}{\partial u \partial v}\right| du dv < +\infty.$$

Denoting $\mathcal{H}_{a,b}^{\otimes 2}$ the second tensor product of $\mathcal{H}_{a,b}$, we also consider $|\mathcal{H}_{a,b}|^{\otimes 2} \subset \mathcal{H}_{a,b}^{\otimes 2}$, the linear space of measurable functions f on $[0,T]$ such that:

$$\|f\|^2_{|\mathcal{H}_{a,b}|^{\otimes 2}} = \int_{[0,T]^4} |f((r,s)f(t,u))|\ \left|\frac{\partial^2 R^{H,a,b}(r,t)}{\partial r \partial t}\frac{\partial^2 R^{H,a,b}(s,u)}{\partial s \partial u}\right| dr ds dt du < +\infty.$$

We denote by $C_b^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that f and all its derivatives are bounded. Let $\mathcal{S}_{a,b}$ be the set of random variables $F_{a,b}$ of the form

$$F_{a,b} = f(Z^H(a,b)(h_1), ..., Z^H(a,b)(h_n)), \tag{3.1}$$

where $f \in C_b^\infty(\mathbb{R}^n)$, h_1, ..., $h_n \in \mathcal{H}_{a,b}$ and $n \geq 1$.

The derivative of a random variable $F_{a,b}$ of the form (3.1) is defined by:

$$D^{a,b}F_{a,b} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Z^H(a,b)(h_1), ..., Z^H(a,b)(h_n))h_i.$$

To make the notation less cluttered, $D^{a,b}$ will be simply denoted D.

The derivative operator is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H}_{a,b})$, for any $p \geq 1$.

We define the iteration of the operator D, in such way that the iterated derivative operator D^k is a random variable with values in $\mathcal{H}_{a,b}^{\otimes k}$, and D^k maps $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H}_{a,b}^{\otimes k})$.

For any positive integer k and any real $p \geq 1$, we denote by $D^{k,p}(\mathcal{H}_{a,b})$, the domain of D^k in $L^p(\Omega)$, which is the closure of $\mathcal{S}_{a,b}$ with respect to the norm defined by

$$\|F\|_{D^{k,p}(\mathcal{H}_{a,b})} = \left[\mathbb{E}(|F|^p) + \mathbb{E}\left(\sum_{j=1}^k \|D^jF\|^p_{\mathcal{H}_{a,b}^{\otimes j}}\right)\right]^{1/p}.$$

In a similar way, given a Hilbert space V we denote by $D^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables.

We denote $D^{1,2}(|\mathcal{H}_{a,b}|)$ the subspace of $D^{1,2}(\mathcal{H}_{a,b})$ consisting of elements u such that $u \in |\mathcal{H}_{a,b}|$ a.s., $Du \in |\mathcal{H}_{a,b}|^{\otimes 2}$ a.s. and

$$\mathbb{E}\left(\|u\|^2_{|\mathcal{H}_{a,b}|} + \|Du\|^2_{|\mathcal{H}_{a,b}|^{\otimes 2}}\right) < \infty.$$

We denote by δ the adjoint of the derivative operator D. That is, δ is an unbounded operator on $L^2(\Omega, \mathcal{H}_{a,b})$ with values in $L^2(\Omega)$ such that:

1) The domain of δ, denoted by $Dom\delta$, is the set of $\mathcal{H}_{a,b}-$ valued square integrable random variables $u \in L^2(\Omega; \mathcal{H}_{a,b})$ such that

$$|\mathbb{E}(< DF, u >_{\mathcal{H}_{a,b}})| \leq c\|F\|_2,$$

for all $F \in D^{1,2}(\mathcal{H}_{a,b})$, where c is some constant depending on u.

2) If $u \in Dom\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(< DF, u >_{\mathcal{H}_{a,b}})$$

for any $F \in D^{1,2}(\mathcal{H}_{a,b})$.

Note that $D^{1,2}(|\mathcal{H}_{a,b}|) \subset D^{1,2}(\mathcal{H}_{a,b}) \subset Dom\delta$. The operator δ is called the divergence operator.

Remark 1. *Denoting $|\mathcal{H}| = |\mathcal{H}_{1,0}|$ the set of measurable functions f on $[0,T]$ such that*

$$\int_0^T \int_0^T |f(u)|\,|f(v)|\,|u-v|^{2H-2} du dv < +\infty,$$

from Lemma 2 we get:

1) $\forall (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\|.\|_{|\mathcal{H}_{a,b}|} \leq C_2 \|.\|_{|\mathcal{H}|}$, $|\mathcal{H}| \subset |\mathcal{H}_{a,b}|$ *and*

$$D^{1,2}(|\mathcal{H}|) \subset D^{1,2}(\mathcal{H}_{a,b}) \subset Dom\delta.$$

2) $\forall (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$; $a \neq b$, $C_1 \|.\|_{|\mathcal{H}|} \leq \|.\|_{|\mathcal{H}_{a,b}|} \leq C_2 \|.\|_{|\mathcal{H}|}$, $|\mathcal{H}| = |\mathcal{H}_{a,b}|$, $D^{1,2}(|\mathcal{H}|) = D^{1,2}(\mathcal{H}_{a,b}) \subset Dom\delta$, *and according to [2], $|\mathcal{H}|$ is a Banach space with respect to both norms $\|.\|_{|\mathcal{H}|}$ and $\|.\|_{|\mathcal{H}_{a,b}|}$. Moreover, the set $\mathcal{E}$, of step functions on $[0;T]$, is dense in $|\mathcal{H}|$.*

The following proposition will be useful. Its proof can be found e.g. in [10].

Proposition 1. *For every $F \in D^{1,2}(\mathcal{H}_{a,b})$ and $u \in Dom\delta$ such that $Fu \in L^2(\Omega; \mathcal{H}_{a,b})$, we have that:*

1) $Fu \in Dom\delta$, *and*
2) $F(\delta u) = \delta(Fu) + < DF, u >_{\mathcal{H}_{a,b}}$.

Now we are able to define a first type integral with respect to the ZgfBm.

Definition 1. *The divergence integral of a process $u \in D^{1,2}(|\mathcal{H}_{a,b}|)$ with respect to $Z^H(a,b)$ is defined by:*

$$\int_0^T u_s \delta(Z_s^H(a,b)) = \delta(u).$$

We can ask in which sense the divergence operator with respect to $Z^H(a,b)$ can be interpreted as a stochastic integral. In order to answer to this question, we will provide a second type of integration with respect to $Z^H(a,b)$. It is about the symmetric integral introduced by Russo and Vallois in [12]. By convention we will assume that all processes and functions vanish outside the interval $[0;T]$.

Definition 2. *Let $u = \{u_t, t \in [0;T]\}$ be a stochastic process with integrable trajectories. The symmetric integral of u with respect to the ZgfBm $Z^H(a,b)$ is defined as the limit in probability as ϵ tends to zero of*

$$\int_0^T u_s \frac{Z_{s+\epsilon}^H(a,b) - Z_{s-\epsilon}^H(a,b)}{2\epsilon} ds \tag{3.2}$$

provided this limit exists, and it is denoted by $\int_0^T u_s dZ_s^H(a,b)$.

In the following theorem, we give a sufficient condition for the existence of the Symmetric integral and we provide the relation between the divergence and the symmetric integrals, which in turn provides a representation of the divergence operator as a stochastic integral.

Theorem 1. *Let $u = \{u_t; t \in [0,T]\}$ be a stochastic process in the space $D^{1,2}(|\mathcal{H}|)$ such that*

$$\int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds dt < \infty \ \ a.s. \tag{3.3}$$

Then, for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the symmetric integral exists and

$$\begin{aligned} \int_0^T u_t dZ_t^H(a,b) &= \int_0^T u_t \delta(Z_t^H(a,b)) \\ &+ \alpha_H \int_0^T \int_0^T D_s u_t \left[(a^2+b^2)|t-s|^{2H-2} - 2ab(t+s)^{2H-2} \right] ds dt \end{aligned} \tag{3.4}$$

with $\alpha_H = H(2H-1)$.

The proof of Theorem 1 follows the same steps as in the proof of Proposition 3 in [2], and it needs the following preliminary lemmas:

Lemma 5. *Let $u = \{u_t; t \in [0,T]\}$ be a stochastic process in the space $D^{1,2}(|\mathcal{H}_{a,b}|)$ and define the approximating process u^ϵ by*

$$u_t^\epsilon := \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} u_s ds, \tag{3.5}$$

where we use the convention $u_s = 0$ for $s \notin [0,T]$. Then, for every $(a,b) \in \mathbb{R}^2$,

1) *when $a \neq b$, there exists a positive constant $e_{H,a,b}$ such that*

$$\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)} \leq e_{H,a,b}\|u\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)}, \tag{3.6}$$

2) *when $a = b$ and $(a,b) \neq (0,0)$, there exists a positive constant $e_{H,a}$ such that*

$$\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)} \leq e_{H,a}\|u\|^2_{D^{1,2}(|\mathcal{H}|)}. \tag{3.7}$$

Proof. From [2], we know that in the particular case of the fractional Brownian motion $B^H = Z^H(1,0)$, there exists a positive constant e_H such that

$$\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{1,0}|)} \leq e_H\|u\|^2_{D^{1,2}(|\mathcal{H}_{1,0}|)}. \tag{3.8}$$

Using Lemma 2 and (3.8), we easily get that

$$\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)} \leq C_4\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{1,0}|)} \leq C_4 e_H\|u\|^2_{D^{1,2}(|\mathcal{H}_{1,0}|)}$$

with $C_4 = \max(C_2, C_2^2)$. Moreover, if $a \neq b$ then, using again Lemma 2, we get

$$\|u^\epsilon\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)} \leq C_4 e_H\|u\|^2_{D^{1,2}(|\mathcal{H}_{1,0}|)} \leq C_5 C_4 e_H\|u\|^2_{D^{1,2}(|\mathcal{H}_{a,b}|)}$$

with $C_5 = \min(C_1, C_1^2)$.

Therefore, Inequalities (3.6) and (3.7) are obtained with $e_{H,a} = C_4 e_H$ and $e_{H,a,b} = C_5 C_4 e_H$.

□

Lemma 6. *Let $u = \{u_t; t \in [0,T]\}$ be a stochastic process in the space $D^{1,2}(|\mathcal{H}|)$. Denoting*

$$A^\epsilon u = \frac{1}{2\epsilon}\int_0^T < Du_s, \mathbf{1}_[s-\epsilon, s+\epsilon] >_{|\mathcal{H}_{a,b}|} ds,$$

for every $\epsilon > 0$, we have

$$\frac{1}{2\epsilon}\int_0^T u_s(Z^H_{s+\epsilon}(a,b) - Z^H_{s-\epsilon}(a,b))ds = \delta(u^\epsilon) + A^\epsilon u, \tag{3.9}$$

and $\delta(u^\epsilon)$ converges in $L^2(\Omega)$ to $\delta(u)$ as ϵ tends to 0.

Proof. Let S_T be the set of smooth step processes of the form:

$$u = \sum_{j=0}^{m-1} F_j \mathbf{1}_{[t_j, t_{j+1}]},$$

where $F_j \in S_{a,b}$, and $0 = t_1 < ... < t_n = T$. We recall that by Remark 1, when $a \neq b$,

$$D^{1,2}(|\mathcal{H}_{a,b}|) = D^{1,2}(|\mathcal{H}|).$$

Since S_T is dense in $D^{1,2}(|\mathcal{H}|)$, there exists a sequence $\{u^n\} \subset S_T$ such that $u^n \to u$ in the norm of the space $D^{1,2}(|\mathcal{H}|)$ as n tends to infinity. Using Assertion 2 in Proposition 1 and applying Fubini's theorem we get

$$\begin{aligned}\frac{1}{2\epsilon}\int_0^T u_s^n(Z_{s+\epsilon}^H(a,b)-Z_{s-\epsilon}^H(a,b))ds &= \frac{1}{2\epsilon}\int_0^T \delta(u_s^n\mathbf{1}_{[}s-\epsilon,s+\epsilon](.))ds + A^\epsilon u^n\\ &= \int_0^T \frac{1}{2\epsilon}\left(\int_{r-\epsilon}^{r+\epsilon} u_s^n ds\right) dZ_r^H(a,b) + A^\epsilon u^n\\ &= \int_0^T u_r^{\epsilon,n} dZ_r^H(a,b) + A^\epsilon u^n\\ &= \delta\left(u_r^{\epsilon,n}\right) + A^\epsilon u^n.\end{aligned}$$

By Lemma 5, we have $u^{\epsilon,n} \to u^\epsilon$ in the norm of the space $D^{1,2}(|\mathcal{H}_{a,b}|)$ as n tends to infinity, which implies that $\delta(u^{\epsilon,n}) \to \delta(u^\epsilon)$ in the norm of the space $L^2(\Omega)$ as n tends to infinity. On another hand, we have

$$\begin{aligned}&\left|\int_0^T (u_s^n - u_s)(Z_{s+\epsilon}^H(a,b)-Z_{s-\epsilon}^H(a,b))ds\right|^2\\ \leq\ & \sup_{|r-s|\leq\epsilon}|Z_r^H(a,b)-Z_s^H(a,b)|^2\left(\int_0^T |u_s^n-u_s|ds\right)^2\\ =\ & \sup_{|r-s|\leq\epsilon}|Z_r^H(a,b)-Z_s^H(a,b)|^2\int_0^T\int_0^T |u_s^n-u_s||u_r^n-u_r|dsdr\\ =\ & \sup_{|r-s|\leq\epsilon}|Z_r^H(a,b)-Z_s^H(a,b)|^2\int_0^T\int_0^T |u_s^n-u_s||u_r^n-u_r|f(r,s)f(r,s)^{-1}dsdr.\end{aligned}$$

where

$$f(r,s) = \begin{cases} \dfrac{\partial^2 R^{H,a,b}(r,s)}{\partial r\partial s} & if \quad a\neq b\\ |r-s|^{2H-2} & if \quad a=b.\end{cases}$$

By Lemma 2, when $a \neq b$,

$$(f(r,s))^{-1} \leq C_1^{-1}\,|r-s|^{2-2H} \leq C_1^{-1}\,T^{2-2H},$$

for every $r,s\in[0,T]; s\neq r$. Thus,

$$\begin{aligned}&\left|\int_0^T (u_s^n - u_s)(Z_{s+\epsilon}^H(a,b)-Z_{s-\epsilon}^H(a,b))ds\right|^2\\ \leq\ & C_1^{-1}T^{2-2H}\sup_{|r-s|\leq\epsilon}|Z_r^H(a,b)-Z_s^H(a,b)|^2\int_0^T\int_0^T |u_s^n-u_s||u_r^n-u_r|f(r,s)dsdr\\ \leq\ & C_1^{-1}C_2T^{2-2H}\sup_{|r-s|\leq\epsilon}|Z_r^H(a,b)-Z_s^H(a,b)|^2\|u^n-u\|^2_{D|^{1,2}(|\mathcal{H}|)}\end{aligned}$$

which converges to zero in probability as n tends to infinity.

In a similar way we can check that $A^\epsilon u^n$ converges in probability to $A^\epsilon u$, which completes the proof of (3.9).

We will now prove that $\delta(u^\epsilon)$ converges in $L^2(\Omega)$ to $\delta(u)$ as ϵ tends to 0. Take a sequence $\{u^n\}\subset S_T$ such that $u^n\to u$ in the norm of the space $D^{1,2}(|\mathcal{H}|)$ as n tends to infinity. For every $n\geq 1$ and $\epsilon>0$ we have

$$\begin{aligned}&\mathbb{E}\left(\delta(u^\epsilon)-\delta(u)\right)^2\\ \leq\ & 3\left[\mathbb{E}\left(\delta(u^\epsilon)-\delta(u^{n,\epsilon})\right)^2+\mathbb{E}\left(\delta(u^{n,\epsilon})-\delta(u^n)\right)^2+\mathbb{E}\left(\delta(u^n)-\delta(u)\right)^2\right].\end{aligned} \tag{3.10}$$

Since $\delta(u^{\epsilon,n})\to\delta(u^\epsilon)$ and $\delta(u^n)\to\delta(u)$ in the norm of the space $L^2(\Omega)$ as n tends to infinity, from (3.10) we get that: for every $\eta>0$, there exists an integer N_η such that, for every integer $n\geq N_\eta$, we have

$$\mathbb{E}\left(\delta(u^\epsilon)-\delta(u)\right)^2 \leq 3\left[\mathbb{E}\left(\delta(u^{n,\epsilon})-\delta(u^n)\right)^2+\eta\right].$$

Moreover, we have $\mathbb{E}\left(\delta(u^{n,\epsilon})-\delta(u^n)\right)^2$ tends to 0 as $\epsilon\to 0$; Consequently, for every $\eta>0$, there exists $\epsilon_0>0$ such that, for every $\epsilon<\epsilon_0$,

$$\mathbb{E}\left(\delta(u^\epsilon)-\delta(u)\right)^2\leq 6\eta,$$

which implies that

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\delta(u^\epsilon)-\delta(u)\right)^2=0.$$

□

Proof. **[of Theorem 1]** By the symmetric integral definition (3.2) and Lemma 6, to get Theorem 1, it suffices to prove that $A^\epsilon u$ converges in probablity to

$$\alpha_H\int_0^T\int_0^T D_s u_t\left[(a^2+b^2)|t-s|^{2H-2}-2ab(t+s)^{2H-2}\right]dsdt$$

when ϵ tends to 0.

For $\epsilon>0$ small enough we have,

$$\begin{aligned} A^\epsilon u &= \frac{1}{2\epsilon}\int_0^T < Du_s,\mathbf{1}_{[s-\epsilon,s+\epsilon]}>_{|\mathcal{H}_{a,b}|}dsdt \\ &= \frac{1}{2\epsilon}\int_0^T\int_0^T D_t u_s\left(\int_{s-\epsilon}^{s+\epsilon}\frac{\partial^2 R^{H,a,b}(r,t)}{\partial r\partial t}dr\right)dsdt \\ &= \frac{1}{2\epsilon}\int_0^T\int_0^T D_t u_s\left(\int_{-\epsilon}^{+\epsilon}\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial\sigma\partial t}d\sigma\right)dsdt. \end{aligned}$$

On the one hand, by the mean value theorem we have

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon}\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial\sigma\partial t}d\sigma=\frac{\partial^2 R^{H,a,b}(s+\sigma_\epsilon,t)}{\partial\sigma\partial t}\tag{3.11}$$

with $\sigma_\epsilon\in(-\epsilon,\epsilon)$. Moreover, the function $\sigma\longmapsto\dfrac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial\sigma\partial t}$ is continuous in 0, for every fixed $0<s<t\leq T$ (see Expression (2.4)). Consequently,

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}D_t u_s\left(\int_{-\epsilon}^{+\epsilon}\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial\sigma\partial t}d\sigma\right)=D_t u_s\frac{\partial^2 R^{H,a,b}(s,t)}{\partial\sigma\partial t}\ a.s..$$

On an other hand, from Lemma 3, there exists a positive constant C_3, such that

$$\frac{|D_t u_s|}{2\epsilon}\int_{-\epsilon}^{\epsilon}\left|\frac{\partial^2 R^{H,a,b}(s+\sigma,t)}{\partial\sigma\partial t}d\sigma\right|d\sigma\leq C_3|D_t u_s||t-s|^{2H-2},\tag{3.12}$$

for every $\epsilon>0$. Thus, using Assumption (3.3) and applying the dominated convergence theorem we get that

$$\lim_{\epsilon\to 0}A^\epsilon u=\int_0^T\int_0^T D_t u_s\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s\partial t}dsdt\ a.s.,$$

which completes the proof of Theorem 1.

□

4 ITÔ FORMULAS WITH ZGFBM

We will establish the following divergence-Integration Itô formula for the ZgfBm $Z^H(a,b)$.

Theorem 2. *Let f be a function of class $C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$, such that,*

$$\sup_{t\in[0,T]} \max\left\{|f(t,x)|, \left|\frac{\partial f(t,x)}{\partial x}\right|, \left|\frac{\partial^2 f(t,x)}{\partial x^2}\right|\right\} \leq c\exp(\lambda x^2), \tag{4.1}$$

for every $x\in\mathbb{R}$, where c and λ are positive constants such that

$$\lambda < \frac{T^{-2H}}{4C_H(a,b)}. \tag{4.2}$$

$$\text{Then,} \qquad f(t,Z_t^H(a,b)) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,Z_s^H(a,b))\delta(Z_s^H(a,b)) + \int_0^t \left(\frac{\partial f}{\partial s}(s,Z_s^H(a,b)) + HC_H(a,b)\frac{\partial^2 f}{\partial x^2}(s,Z_s^H(a,b))s^{2H-1}\right)ds$$

where $C_H(a,b)$ is the constant defined in (2.3).

Proof. Let us fix $t>0$ and

$$\sigma := \left\{t_0 = 0 < t_1 = \frac{t}{n} < \ldots < t_{n-1} = \frac{(n-1)t}{n} < t_n = t\right\}$$

a partition of the interval $[0,t]$.

By the localization argument and the fact that the process $Z^H(a,b)$ is continuous, we can assume that f has compact support, and so there exists a positive constant C majorizing f and all its partial derivatives.

Using Taylor expansion , we get

$$\begin{aligned} & f(t,Z_t^H(a,b)) = f(0,0) + \sum_{j=1}^n \Big(f(t_j,Z_{t_j}^H(a,b)) - f(t_{j-1},Z_{t_{j-1}}^H(a,b))\Big) \\ = \; & f(0,0) + \sum_{j=1}^n (t_j - t_{j-1})\frac{\partial f}{\partial s}(X_{j-1}) + \sum_{j=1}^n (Z_{t_j}^H(a,b) - Z_{t_{j-1}}^H(a,b))\frac{\partial f}{\partial x}(X_{j-1}) \\ & + \frac{1}{2}\sum_{j=1}^n (t_j - t_{j-1})^2 \frac{\partial^2 f}{\partial s^2}(\tilde{X}_j) + \frac{1}{2}\sum_{j=1}^n (Z_{t_j}^H(a,b) - Z_{t_{j-1}}^H(a,b))^2 \frac{\partial^2 f}{\partial x^2}(\tilde{X}_j) \\ & + \sum_{j=1}^n (t_j - t_{j-1})(Z_{t_j}^H(a,b) - Z_{t_{j-1}}^H(a,b))\frac{\partial^2 f}{\partial s\partial x}(\tilde{X}_j) \\ = \; & f(0,0) + \sum_{p=1}^5 I_{n,p}, \end{aligned}$$

with $X_j = (t_j, Z_{t_j}^H)$, $\tilde{X}_j = X_{j-1} + \theta_j(X_j - X_{j-1})$, and θ_j is a random variable in $(0,1)$.

Using Cauchy Schwarz inequality, the fact that the increments $Z_{t_j}^H(a,b) - Z_{t_{j-1}}^H(a,b)$ are centered Gaussian random variables, the second inequality in (2.2) and the fact that $\frac{1}{2} < H < 1$ we get:

$$\begin{aligned}
\mathbb{E}(I_{n,5}^2) &= \mathbb{E}\left(\sum_{j=1}^{n}(t_j-t_{j-1})(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))\frac{\partial^2 f}{\partial s\partial x}(\tilde{X}_j)\right)^2 \\
&\leq \sum_{j=1}^{n}\mathbb{E}\left((t_j-t_{j-1})^2(\frac{\partial^2 f}{\partial s\partial x}(\tilde{X}_j))^2\right)\sum_{j=1}^{n}\mathbb{E}\left(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b)\right)^2 \\
&\leq C^2\sum_{j=1}^{n}(t_j-t_{j-1})^2\sum_{j=1}^{n}\mathbb{E}\left(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b)\right)^2 \\
&\leq C^2\frac{t^2}{n}\sum_{j=1}^{n}\mathbb{E}\left(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b)\right)^2 \\
&\leq \nu(a,b,H)C^2\frac{t^2}{n}\sum_{j=1}^{n}(t_j-t_{j-1})^{2H} \\
&\leq \nu(a,b,H)C^2\frac{t^{2H+2}}{n^{2H}} \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(I_{n,4}^2) &= \frac{1}{4}\mathbb{E}\left(\sum_{j=1}^{n}(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))^2\frac{\partial^2 f}{\partial x^2}(\tilde{X}_j)\right)^2 \\
&\leq \frac{1}{4}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\partial^2 f}{\partial x^2}(\tilde{X}_j)\right)^2\sum_{j=1}^{n}\mathbb{E}\left((Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))^4\right) \\
&\leq \frac{nC^2}{4}\sum_{j=1}^{n}\mathbb{E}\left((Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))^4\right) \\
&= 3\frac{nC^2}{4}\sum_{j=1}^{n}\left(\mathbb{E}(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))^2\right)^2 \\
&\leq 3\nu^2(a,b,H)\frac{nC^2}{4}\sum_{j=1}^{n}(t_j-t_{j-1})^{4H} \\
&= 3\nu^2(a,b,H)\frac{n^{2-4H}C^2}{4}t^{4H} \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(I_{n,3}^2) &= \mathbb{E}\left(\sum_{j=1}^{n}(t_j-t_{j-1})^2\frac{\partial^2 f}{\partial s^2}(\tilde{X}_j)\right)^2 \\
&\leq \sum_{j=1}^{n}(t_j-t_{j-1})^4\sum_{j=1}^{n}\mathbb{E}\left((\frac{\partial^2 f}{\partial s^2}(\tilde{X}_j))^2\right) \\
&\leq nC^2\sum_{j=1}^{n}(t_j-t_{j-1})^4 \\
&= \frac{C^2t^4}{n^2} \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Now, in order to study the limit of the sequence $(I_{n,2})$, we will apply Proposition 1. Let us first note that, using Lemma 4, Conditions (4.1) and (4.2) imply that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|f(t,Z_t^H(a,b))\right|^2\right]\le c^2\mathbb{E}\left[\exp\left(2\lambda\sup_{0\le t\le T}\left|Z_t^H(a,b)\right|^2\right)\right]<\infty.$$

In a similar way we get that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\partial f(t,Z_t^H(a,b))}{\partial x}\right|^2\right]<\infty \; and \; \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\partial^2 f(t,Z_t^H(a,b))}{\partial x^2}\right|^2\right]<\infty.$$

Therefore, under the growth condition (4.1), the process $\frac{\partial f}{\partial x}(t,Z_t^H(a,b))$ belongs to the space $D^{1,2}(|\mathcal{H}_{a,b}|)$ and (3.3) holds. Applying Proposition 1 we get

$$\begin{aligned}
I_{n,2} &= \sum_{j=1}^n \frac{\partial f}{\partial x}(t_{j-1},Z_{t_{j-1}}^H(a,b))(Z_{t_j}^H(a,b)-Z_{t_{j-1}}^H(a,b))\\
&= \sum_{j=1}^n \frac{\partial f}{\partial x}(t_{j-1},Z_{t_{j-1}}^H(a,b))\delta\left(\mathbf{1}_{(t_{j-1},t_j]}\right)\\
&= \delta\left(\sum_{j=1}^n \frac{\partial f}{\partial x}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left(\mathbf{1}_{(t_{j-1},t_j]}\right)\right)\\
&\quad +\sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left\langle \mathbf{1}_{(0,t_{j-1}]},\mathbf{1}_{(t_{j-1},t_j]}\right\rangle_{\mathcal{H}_{a,b}}\\
&= I_{n,2,1}+I_{n,2,2}.
\end{aligned}$$

We have

$$\begin{aligned}
I_{n,2,2} &= \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left\langle \mathbf{1}_{(0,t_{j-1}]},\mathbf{1}_{(t_{j-1},t_j]}\right\rangle_{\mathcal{H}_{a,b}}\\
&= \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left(\left\langle \mathbf{1}_{(0,t_{j-1}]},\mathbf{1}_{0,t_j]}\right\rangle_{\mathcal{H}_{a,b}}-\left\langle \mathbf{1}_{(0,t_{j-1}]},\mathbf{1}_{(0,t_{j-1}]}\right\rangle_{\mathcal{H}_{a,b}}\right)\\
&= \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left(R^{H,a,b}(t_{j-1},t_j)-R^{H,a,b}(t_{j-1},t_{j-1})\right)\\
&= \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left(\frac{1}{2}(a+b)^2(t_{j-1}^{2H}+t_j^{2H})-ab(t_j+t_{j-1})^{2H}\right.\\
&\quad \left.-\frac{a^2+b^2}{2}\mid t_j-t_{j-1}\mid^{2H}\right)-\sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left((a+b)^2-2^{2H}ab\right)t_{j-1}^{2H}\\
&= \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1},Z_{t_{j-1}}^H(a,b))\left(\frac{1}{2}(a+b)^2(t_{j-1}^{2H}+t_j^{2H})-ab(t_j+t_{j-1})^{2H}\right.\\
&\quad \left.-\frac{a^2+b^2}{2}\mid t_j-t_{j-1}\mid^{2H}-\left((a+b)^2-2^{2H}ab\right)t_{j-1}^{2H}\right).
\end{aligned}$$

We will prove that $(I_{n,2,2})$ converges, in $L^2(\Omega)$, to

$$\int_0^t \frac{\partial^2 f}{\partial x^2}(s,Z_s^H(a,b))H\left[a^2+b^2-(2^{2H}-2)ab\right]s^{2H-1}ds.$$

Denoting

$$A_t = \int_0^t H\left[(a^2+b^2-(2^{2H}-2)ab\right]s^{2H-1}ds = \frac{1}{2}\left[a^2+b^2-(2^{2H}-2)ab\right]t^{2H},$$

to touch our target, it suffices to prove that the sequence (C_n) defined by

$$C_n =: \left[\mathbb{E}\left(I_{n,2,2} - \sum_{j=1}^n \frac{\partial^2 f}{\partial x^2}(t_{j-1}, Z^H_{t_{j-1}}(a,b))(A_{t_j}-A_{t_{j-1}})\right)^2\right]^{1/2},$$

converges to 0, as n tends to ∞.

Again by Cauchy-Schwarz inequality we get

$$\begin{aligned}
C_n &\leq \sum_{j=1}^n \sqrt{\mathbb{E}\left(\frac{\partial^2 f}{\partial x^2}(t_{j-1}, Z^H_{t_{j-1}}(a,b))\right)^2}\left|\frac{1}{2}(a+b)^2(t_{j-1}^{2H}+t_j^{2H}) - ab(t_j+t_{j-1})^{2H}\right.\\
&\quad -\frac{a^2+b^2}{2}\mid t_j - t_{j-1}\mid^{2H} - \left((a+b)^2-2^{2H}ab\right)t_{j-1}^{2H}\\
&\quad \left.-\frac{1}{2}\left[a^2+b^2-(2^{2H}-2)ab\right](t_j^{2H}-t_{j-1}^{2H})\right|\\
&\leq C\frac{t^{2H}}{n^{2H}}\sum_{j=1}^n\left|\frac{1}{2}(a+b)^2((j-1)^{2H}+j^{2H}) - ab(2j-1)^{2H}\right.\\
&\quad -\frac{a^2+b^2}{2} - \left((a+b)^2-2^{2H}ab\right)(j-1)^{2H}\\
&\quad \left.-\frac{1}{2}\left[a^2+b^2-(2^{2H}-2)ab\right](j^{2H}-(j-1)^{2H})\right|\\
&\leq C\frac{t^{2H}}{n^{2H}}\sum_{j=1}^n\left|2^{2H}ab\left[\frac{(j-1)^{2H}+j^{2H}}{2} - \left(\frac{2j-1}{2}\right)^{2H}\right] - \frac{a^2+b^2}{2}\right|\\
&\leq C\frac{t^{2H}}{n^{2H}}\sum_{j=1}^n\left|2^{2H}ab\left[\frac{(j-1)^{2H}+j^{2H}}{2} - \left(\frac{2j-1}{2}\right)^{2H}\right]\right| + C\frac{a^2+b^2}{2}\frac{t^{2H}}{n^{2H-1}}\\
&= Cab\frac{t^{2H}}{n^{2H}}\sum_{j=1}^n |h(j)| + C\frac{a^2+b^2}{2}\frac{t^{2H}}{n^{2H-1}}
\end{aligned}$$

with $h(j) = \dfrac{(2j-2)^{2H}+(2j)^{2H}}{2} - (2j-1)^{2H} = \dfrac{g(2j-1)}{2}$, where g is the function defined by:

$$g : x \longmapsto (x+1)^{2H} - 2x^{2H} + (x-1)^{2H}.$$

On the one hand, since $H > 1/2$ we have $\lim_{n\to+\infty} C\dfrac{a^2+b^2}{2}\dfrac{t^{2H}}{n^{2H-1}} = 0.$

On the other hand, g is differentiable and

$$g'(x) = 2H\left[(x+1)^{2H-1} - 2x^{2H-1} + (x-1)^{2H-1}\right]$$

for evey $x > 0$. Since $H > 1/2$, the function $x \longmapsto x^{2H-1}$ is concave, and therefore,

$$x^{2H-1} \geq \frac{1}{2}\left[(x+1)^{2H-1} + (x-1)^{2H-1}\right],$$

which implies that $g'(x) \leq 0$ for every $x > 0$. Thus g is a non increasing function, and consequently

$$h(j) = \frac{g(2j-1)}{2} \leq \frac{g(1)}{2} = 2^{2H-1} - 1$$

for every $1 \leq j \leq n$. Moreover, since the function $x \longmapsto x^{2H}$ is convex, $g(x) \geq 0$ for every $x > 0$, and consequently, $h(j) = \dfrac{g(2j-1)}{2} \geq 0$ for every $j \geq 1$.

So, $|Cab\dfrac{t^{2H}}{n^{2H}}\sum_{j=1}^{n} h(j)| \leq C|ab|\dfrac{t^{2H}}{n^{2H-1}}(2^{2H-1}-1) \to 0$ as $n \to \infty$.

We will now show that $(I_{n,2,1})$ converges to

$$\delta\left(\frac{\partial f}{\partial x}(., Z^H_{.}(a,b))\mathbf{1}_{(0,t)}(.)\right) = \int_0^t \frac{\partial f}{\partial x}(s, Z^H_s(a,b))\delta(Z^H_s(a,b))$$

in $L^2(\Omega)$. For this, we will first show that $\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b))\left(\mathbf{1}_{(t_{j-1},t_j]}\right)\right)$ converges to $\frac{\partial f}{\partial x}(., Z^H_{.}(a,b))\mathbf{1}_{(0,t)(.)}$ in $L^2(\Omega; \mathcal{H}_{a,b})$. We have,

$$\begin{aligned}
& \mathbb{E}\left(\left\|\sum_{j=1}^{n}[\frac{\partial f}{\partial x}(t_{j-1}(a,b), Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(., Z^H_{.}(a,b))]\mathbf{1}_{(t_{j-1},t_j]}(.)\right\|^2_{|\mathcal{H}_{a,b}|}\right) \\
= & \mathbb{E}\left(\sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} |\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))|\right. \\
& \left.\times\ |\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(v, Z^H_v(a,b))|\frac{\partial^2 R^{H,a,b}}{\partial u \partial v}(u,v)dudv\right). \\
= & \sum_{i,j=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} \mathbb{E}\left(|\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))|\right. \\
& \left.\times|\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(v, Z^H_v(a,b))|\right)\ \frac{\partial^2 R^{H,a,b}}{\partial u \partial v}(u,v)dudv.
\end{aligned}$$

Applying Cauchy Schwarz Inequality, we get

$$\begin{aligned}
& \mathbb{E}\left(\left|\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))\right|\right. \\
& \left.\times\ \left|\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(v, Z^H_v(a,b))\right|\right) \\
\leq & \sqrt{\mathbb{E}\left(\left|\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))\right|^2\right)} \\
& \times\sqrt{\mathbb{E}\left(\left|\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(v, Z^H_v(a,b))\right|^2\right)}.
\end{aligned}$$

By the Mean Value Theorem we get

$$\begin{aligned}
& \frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b)) \\
= & \frac{\partial^2 f}{\partial x^2}(Y_{i-1})(Z^H_{t_{i-1}}(a,b) - Z^H_u(a,b)) + \frac{\partial^2 f}{\partial s \partial x}(Y_{i-1})(t_{i-1} - u),
\end{aligned}$$

where $Y_{i-1} = (1-\alpha)(t_{i-1}, Z^H_{t_{i-1}}(a,b)) + \alpha(u, Z^H_u(a,b))$, with α is a random variable in $(0,1)$.

Thus, by (2.2), we get

$$\begin{aligned}
&\mathbb{E}\left(|\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))|^2\right) \\
\leq\ & 2C\left(\mathbb{E}(Z^H_{t_{i-1}}(a,b) - Z^H_u(a,b))^2 + (t_{i-1}-u)^2\right) \\
\leq\ & 2C\left(\nu(a,b,H)|t_{i-1}-u|^{2H} + (t_{i-1}-u)^2\right) \\
\leq\ & 2C(\nu(a,b,H) + T^{2-2H})|t_{i-1}-u|^{2H}
\end{aligned}$$

for every $u \in (t_{i-1}, t_i)$, and $i \in \{1, ..., n\}$.

Therefore

$$\begin{aligned}
&\sqrt{\mathbb{E}\left(\left|\frac{\partial f}{\partial x}(t_{i-1}, Z^H_{t_{i-1}}(a,b)) - \frac{\partial f}{\partial x}(u, Z^H_u(a,b))\right|^2\right)} \\
&\times\sqrt{\mathbb{E}\left(\left|\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(v, Z^H_v(a,b))\right|^2\right)} \\
\leq\ & Cte\ |t_{i-1}-u|^H |t_{j-1}-v|^H \\
\leq\ & Cte\ \sup_{i=1}^{n} |t_{i-1}-t_i|^{2H}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left(\left\|\sum_{j=1}^{n}[\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(., Z^H_.(a,b))]\mathbf{1}_{(t_{j-1},t_j]}(.)\right\|^2_{|\mathcal{H}_{a,b}|}\right) \\
\leq\ & Cte\ \sup_{i=1}^{n} |t_{i-1}-t_i|^{2H} \sum_{i,j=1}^{n} \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} \frac{\partial^2 R^{H,a,b}}{\partial u \partial v}(u,v)dudv \\
\leq\ & Cte\ \sup_{i=1}^{n} |t_{i-1}-t_i|^{2H} R^{H,a,b}(t,t) \\
=\ & Cte\ \left(\frac{t}{n}\right)^{2H} R^{H,a,b}(t,t).
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\mathbb{E}\left(\left\|\sum_{j=1}^{n}[\frac{\partial f}{\partial x}(t_{j-1}, Z^H_{t_{j-1}}(a,b)) - \frac{\partial f}{\partial x}(., Z^H_.(a,b))]\mathbf{1}_{(t_{j-1},t_j]}(.)\right\|^2_{|\mathcal{H}_{a,b}|}\right) = 0.$$

All this, with the fact that $\lim_{n\to\infty} I_{n,1} = \int_0^t \frac{\partial f}{\partial s}(s, Z^H_s(a,b))ds\ a.s.$ allows us to finish the proof. □

In the particular case where the function f of Theorem 2 is independent of time we get the following Malliavin divergence and Stochastic symmetric forms of Itô formulas.

Corollary 1. *Let f be a function of class $C^2(\mathbb{R})$ such that,*

$$\max\left\{|f(x)|, |f'(x)|, |f''(x)|\right\} \leq c\exp(\lambda x^2), \tag{4.3}$$

for every $x \in \mathbb{R}$, where c and λ are positive constants satisfying Condition (4.2). Then

$$\begin{aligned}
&f(Z^H_t(a,b)) = f(0) + \int_0^t f'(Z^H_s(a,b))\delta(Z^H_s(a,b)) \\
+\ & \int_0^t HC_H(a,b)f''(Z^H_s(a,b))s^{2H-1}ds \\
=\ & f(0) + \int_0^t f'(Z^H_s(a,b))dZ^H_s(a,b) \\
+\ & \int_0^t f''(Z^H_s(a,b))H\left[2ab(t+s)^{2H-1} - 2^{2H}abs^{2H-1} - (a^2+b^2)(t-s)^{2H-1}\right]ds.
\end{aligned} \tag{4.4}$$

Proof. The first equality in (4.4) is clearly due to Theorem 2. Moreover, Assumptions (4.3) and (4.2) imply that the stochastic process $u_t := \frac{\partial f}{\partial x}(t, Z_t^H(a,b)) \in D^{1,2}(|\mathcal{H}_{a,b}|)$ and (u_t) satisfies Condition (3.3). Consequently, Theorem 1 yields

$$\begin{aligned} f(Z_t^H(a,b)) &= f(0) + \int_0^t f'(Z_s^H(a,b))dZ_s^H(a,b) \\ &- \int_0^t \int_0^t D_r(f'(Z_s^H(a,b))\frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dsdr \\ &+ \int_0^t HC_H(a,b)f''(Z_s^H(a,b))s^{2H-1}ds. \end{aligned} \tag{4.5}$$

On the one hand,

$$\int_0^t \int_0^t D_r(f'(Z_s^H(a,b))\frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dsdr = \int_0^t f''(Z_s^H(a,b)) \left(\int_0^t \frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dr \right) ds, \tag{4.6}$$

and, on another hand, for every $0 \le s \le t$, using Expression (2.4), we have

$$\begin{aligned} \int_0^t \frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dr &= \int_0^s \frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dr + \int_s^t \frac{\partial^2 R^{H,a,b}(s,r)}{\partial s \partial r} dr \\ &= \int_0^s H(2H-1) \left[(a^2+b^2)(s-r)^{2H-2} - 2ab(s+r)^{2H-2} \right] dr \\ &+ \int_s^t H(2H-1) \left[(a^2+b^2)(r-s)^{2H-2} - 2ab(r+s)^{2H-2} \right] dr \\ &= H \left[s^{2H-1}(a+b)^2 + (a^2+b^2)(t-s)^{2H-1} - 2ab(t+s)^{2H-1} \right]. \end{aligned}$$

All this yields the second equality in (4.4). □

Remark 2. *Applying the first Equality in Formula (4.4) with $a = b = \frac{1}{\sqrt{2}}$, we retrieve the Itô Formula with sfBm obtained in [17]. When $a = 1$ and $b = 0$, we retrieve the formula with fBm given e.g. in [2].*

We will now provide some examples of application of the established Itô formulas.

4.1 Example 1

Considering $f(x) = \frac{x^2}{2}$, for every $x \in \mathbb{R}$ we have

$$\max \left(|f(x)|, |f'(x)|, |f''(x)| \right) = \max \left(\frac{x^2}{2}, |x|, 1 \right) \le ce^{\lambda x^2},$$

for every $0 < \lambda$ and $c = \max\left(\frac{1}{2\lambda}, 1\right)$. Therefore, Condition (4.3) is satisfied, and as consequence, applying Corollary 1, we get

$$\int_0^t Z_s^H(a,b)\delta(Z_s^H(a,b)) = \frac{1}{2}\left(Z_s^H(a,b)\right)^2 - H\left[(a^2+b^2-(2^{2H}-2)ab\right]t^{2H}.$$

4.2 Example 2

Considering $f(t,x) = tx$, for every $x \in \mathbb{R}$ we have

$$\sup_{t \in [0,T]} \max \left\{ |f(t,x)|, \left| \frac{\partial f(t,x)}{\partial x} \right|, \left| \frac{\partial^2 f(t,x)}{\partial x^2} \right| \right\} = T \max(|x|, 1) \le ce^{\lambda x^2},$$

for every $0 < \lambda$ and $c = T\max\left(\frac{1}{\sqrt{\lambda}}, 1\right)$. Therefore, Condition (4.1) is satisfied, and as consequence, applying Theorem 2 we get

$$\int_0^t s\delta(Z_s^H(a,b)) = tZ_t^H(a,b) - \int_0^t Z_s^H(a,b)ds.$$

Now we will provide a third interesting application of Theorem 2, via which we introduce and resolve a generalized version of the known Fractional Black-Scholes Equation.

4.3 Example 3: *Generalized Fractional Black-Scholes equation*

We introduce a generalized version of the fractional Black–Scholes process (ZgfBSp) that we define by:

$$S_t = S_0 \exp\left(\int_0^t (\nu(s) - \sigma^2 s^{2H-1})ds + \sigma Z_t^H(a,b)\right).$$

In the particular case where $a = 1$ and $b = 0$, the ZgfBSp is no other than the classical fractional Black–Scholes process, that satisfies the option pricing model:

$$\delta S_t = S_t \nu(t)dt + \sigma S_t \delta B_t^H, \tag{4.7}$$

where B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, ν is a deterministic function and σ is a strictly positive constant. For more details on the classical fractional Black–Scholes process see e.g. [14].

Considering $f(t,x) = \psi(t)e^{\sigma x}$ with

$$\psi(t) = \exp\left(\int_0^t (\nu(s) - \sigma^2 H\left[a^2 + b^2 - (2^{2H} - 2)ab\right] s^{2H-1})ds\right),$$

we have

$$\sup_{t\in[0,T]} \max\left\{|f(t,x)|, \left|\frac{\partial f(t,x)}{\partial x}\right|, \left|\frac{\partial^2 f(t,x)}{\partial x^2}\right|\right\} \leq C_T e^{\sigma|x|},$$

with

$$C_T = \exp\left(\int_0^T (|\nu(s)| + \sigma^2 H\left|a^2 + b^2 - (2^{2H} - 2)ab\right| s^{2H-1})ds\right).$$

Since, for every $x \in \mathbb{R}$, we have

$$e^{\sigma|x|} \leq \begin{cases} e^{\lambda x^2} & if \quad |x| \geq \frac{\sigma}{\lambda} \\ e^{\frac{\sigma^2}{\lambda}} e^{\lambda x^2} & if \quad |x| < \frac{\sigma}{\lambda} \end{cases},$$

we get $C_T e^{\sigma x} \leq c e^{\lambda x^2}$, for every $\lambda > 0$, with $c = C_T \max(1, e^{\sigma^2/\lambda})$.

Therefore, Condition (4.1) is satisfied, and applying Theorem 2 we get the following generalized version of the fractional Black–Scholes pricing option model (4.7):

$$\delta S_t = S_t \nu(t)dt + \sigma S_t \delta Z_t^H(a,b). \tag{4.8}$$

ACKNOWLEDGMENT

The author wishes to thank the referees and editors for their valuable comments and suggestions which led to improvements in the document.

DECLARATION

The author declares no conflict of interest.

REFERENCES

[1] Alòs, E., Mazet, O., & Nualart, D. (2001). Stochastic calculus with respect to Gaussian processes. *The Annals of Probability*, 29(2), 766-801.

[2] Alòs, E., & Nualart, D. (2003). Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastic Reports*, 75(3), 129-152.

[3] Bojdecki, T., Gorostiza, L. G., & Talarczyk, A. (2004). Sub-fractional Brownian motion and its relation to occupation times. *Statistics & Probability Letters*, 69(4), 405-419.

[4] E. Nouty, C. & Zili, M. (2015). On the sub-mixed fractional Brownian motion. *Applied Mathematics-A Journal of Chinese Universities*, 30, 27-43.

[5] Houdrè, C., & Villa, J. (2003). An example of infinite dimensional quasi-helix. *Contemporary Mathematics*, 336, 195-202.

[6] Mandelbrot, B. B., & Van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM review*, 10(4), 422-437.

[7] Mishura, Y., & Mishura, I. S. (2008). *Stochastic calculus for fractional Brownian motion and related processes* (Vol. 1929). Springer Science & Business Media.

[8] Mishura, Y., & Zili, M. (2018). *Stochastic analysis of mixed fractional Gaussian processes*. Elsevier.

[9] Nourdin, I. (2012). *Selected aspects of fractional Brownian motion* (Vol. 4). Milan: Springer.

[10] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer-Verlag Berlin Heidelberg, (2006).

[11] Peltier, R. F., & Véhel, J. L. (1995). *Multifractional Brownian motion: definition and preliminary results* (Doctoral dissertation, INRIA).

[12] Russo, F., & Vallois, P. (1993). Forward, backward and symmetric stochastic integration. *Probability theory and related fields*, 97, 403-421.

[13] Sghir, A. (2013). The generalized sub-fractional Brownian motion. *Communications on Stochastic Analysis*, 7(3), 2.

[14] Sottinen, T., & Valkeila, E. (2003). On arbitrage and replication in the fractional Black–Scholes pricing model. *Statistics & Decisions*, 21(2), 93-108.

[15] Tudor, C. (2007). Some properties of the sub-fractional Brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 79(5), 431-448.

[16] Tudor, C. (2008). Some aspects of stochastic calculus for the sub-fractional Brownian motion. *Ann. Univ. Bucuresti, Mathematica*, 199-230.

[17] Yan, L., Shen, G., & He, K. (2011). Itô's formula for a sub-fractional Brownian motion. *Communications on Stochastic Analysis*, 5(1), 9.

[18] Zili, M. (2017). Generalized fractional Brownian motion. *Modern Stochastics: Theory and Applications*, 4(1), 15-24.

[19] Zili, M. (2018). On the generalized fractional Brownian motion. *Mathematical Models and Computer Simulations*, 10(6), 759-769.

[20] Zili, M. (2006). On the mixed fractional Brownian motion. *International Journal of stochastic analysis*, 2006.

[21] Zili, M. (2014). Mixed sub-fractional Brownian motion. *Random Operators and Stochastic Equations*, 22(3), 163-178.