IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

Representation of Dimant strongly (p, σ)-continuous multilinear operators by trace duality

Elhadj Dahia[1]

ABSTRACT: We introduce a tensor norm which represents the space of Dimant strongly (p, σ)-continuous multilinear operators by trace duality.

Keywords: Tensorial representation, tensor norm, trace duality.

MSC: Primary 46A32; Secondary 47B10

1 Introduction and preliminaries.

The concept of (p, σ)-absolutely continuous linear operators, was introduced by Matter [10], in order to analyze super-reflexive Banach spaces, establishing many of its fundamental properties. In the nineties, this concept developed by López Molina and Sánchez Pérez [8]. The class of (p, σ)-absolutely continuous operators can be considered as an "interpolated" class between the p-summing operators and the continuous operators, preserving some of the characteristic properties of the first class.

In 2013 Dahia et al. [6] defined and characterized the class of $(p; p_1, ..., p_m; \sigma)$-absolutely continuous multilinear operators on Banach spaces as a natural multilinear extension of the classical class of (p, σ)-absolutely continuous linear operators and extends almost all the ones that are satisfied by the class of absolutely p-summing and p -dominated multilinear operators. On the other hand, the class of all Dimant strongly (p, σ)-continuous multilinear operators was introduced by Achour et al. in [1] as an intermediate class between the class of strongly multilinear operators (see [7]) and the class of all continuous multilinear operators.

In this paper, we present a tensor norm that satisfies that the topological dual of the corresponding normed tensor product is isometric to the space of all Dimant strongly (p, σ)-continuous multilinear operators. Note that the idea of tensorial representation has worked successfully in many subclass of multilinear operators (see [2], [3], [4], [5], [6], [9] and the references therein).

Let $m \in \mathbb{N}$ and $X_j, (j = 1, ..., m), Y$ be Banach spaces over $\mathbb{K}$, (either $\mathbb{R}$ or $\mathbb{C}$). We will denote by $\mathcal{L}(X_1, ..., X_m; Y)$ the Banach space of all continuous m-linear mappings from $X_1 \times ... \times X_m$ into Y, under the norm

$$\|T\| = \sup_{x_j \in B_{X_j}, 1 \leq j \leq m} \left\| T(x^1, ..., x^m) \right\|,$$

- [1]*Elhadj Dahia, corresponding author, Laboratoire de Mathématiques et Physique Appliquées, École Normale Supérieure de Bousaada 28001 Bousaada, Algeria; and Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, University of M'sila, Algeria.*
E-mail: dahia.elhadj@ens-bousaada.dz

Communicated Editor: Guediri Hocine
Manuscript received Dec 26, 2023; revised Jan 28, 2024; accepted Feb 03, 2024; published May 05, 2024.

where B_{X_j} denotes the closed unit ball of $X_j(1 \leq j \leq m)$. Let now X be a Banach space and $1 \leq p < \infty$. We write p^* for the real number satisfying $1/p + 1/p^* = 1$. We denote by $\ell_p^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

and by $\ell_{p,\omega}^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_{p,\omega} = \sup_{\|\phi\|_{X^*}\leq 1} \left(\sum_{i=1}^n |\langle x_i, \phi\rangle|^p\right)^{\frac{1}{p}},$$

where X^* denotes the topological dual of X.

Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. For all $(x_i^j)_{i=1}^n \subset X_j, (1 \leq j \leq m)$ we put

$$\delta_{p\sigma}((x_i^j)_{i=1}^n) = \sup_{\varphi\in B_{\mathcal{L}(X_1,\ldots,X_m)}} \left(\sum_{i=1}^n \left(\left|\varphi(x_i^1,\ldots,x_i^m)\right|^{1-\sigma} \prod_{j=1}^m \left\|x_i^j\right\|^\sigma\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

It is clear that

$$\sup_{\varphi\in B_{\mathcal{L}(X_1,\ldots,X_m)}} \left(\sum_{i=1}^n \left|\varphi(x_i^1,\ldots,x_i^m)\right|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \leq \delta_{p\sigma}((x_i^j)_{i=1}^n),$$

for all $(x_i^j)_{i=1}^n \subset X_j$, $1 \leq j \leq m$.

Definition 1.1. A mapping $T \in \mathcal{L}(X_1,\ldots,X_m;Y)$ is Dimant strongly (p,σ)-continuous if there is a constant $C > 0$ such that for any $x_1^j,\ldots,x_n^j \in X_j$, $1 \leq j \leq m$, we have

$$\left\|\left(T\left(x_i^1,\ldots,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \leq C\ \delta_{p\sigma}((x_i^j)_{i=1}^n). \tag{1.1}$$

The class of all Dimant strongly (p,σ)-continuous m-linear operators from $X_1 \times \cdots \times X_m$ into Y, which is denoted by $\mathcal{L}_p^{s,\sigma}(X_1,\ldots,X_m;Y)$ is a Banach space with the norm $\|T\|_{\mathcal{L}_p^{s,\sigma}}$ which is the smallest constant C such that the inequality (1.1) holds.

2 Tensorial representation

We introduce a tensor norm on $X_1 \otimes ... \otimes X_m \otimes Y$ so that the topological dual of the resulting space is isometric to $(\mathcal{L}_p^{s,\sigma}(X_1,...,X_m;Y^*), \|\cdot\|_{\mathcal{L}_p^{s,\sigma}})$.

The injective tensor norm on $X_1 \otimes ... \otimes X_m \otimes Y$ is defined by

$$\epsilon(u) = \sup_{\phi_j\in B_{X_j^*},\phi\in B_{Y^*}} \left|\sum_{i=1}^n \phi_1(x_i^1)...\phi_m(x_i^m)\phi(y_i)\right|,$$

where $\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ is any representation of $u \in X_1 \otimes ... \otimes X_m \otimes Y$. The projective tensor norm on $X_1 \otimes ... \otimes X_m \otimes Y$ is defined by

$$\pi(u) = \inf \sum_{i=1}^n \left\|x_i^1\right\| ... \|x_i^m\| \|y_i\|,$$

where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ with $x_i^j \in X_j, y_i \in Y, i = 1,...,n, j = 1,...,m$.

For $1 \leq p, r < \infty, 0 \leq \sigma < 1$ with $\frac{1}{r} + \frac{1-\sigma}{p} = 1$ and $u \in X_1 \otimes ... \otimes X_m \otimes Y$, we consider

$$d_{p,\sigma}(u) = \inf \delta_{p\sigma}((x_i^j)_{i=1}^n)\, \|(y_i)_{i=1}^n\|_r,$$

where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$.

Proposition 2.1. *$d_{p,\sigma}$ is a reasonable crossnorm on $X_1 \otimes ... \otimes X_m \otimes Y$ and $\epsilon \leq d_{p,\sigma} \leq \pi$.*

Proof. Let $u', u'' \in X_1 \otimes ... \otimes X_m \otimes Y$. For all $\varepsilon > 0$ choose representations of u' and u'' of the form

$$u' = \sum_{i=1}^{n'} x_i'^1 \otimes ... \otimes x_i'^m \otimes y_i', \quad u'' = \sum_{i=1}^{n''} x_i''^1 \otimes ... \otimes x_i''^m \otimes y_i'',$$

such that

$$d_{p,\sigma}(u') + \varepsilon \geq \delta_{p\sigma}((x_i'^j)_{i=1}^{n'}). \left\| (y_i')_{i=1}^{n'} \right\|_r \quad \text{and} \quad d_{p,\sigma}(u'') + \varepsilon \geq \delta_{p\sigma}((x_i''^j)_{i=1}^{n''}). \left\| (y_i'')_{i=1}^{n''} \right\|_r .$$

We can write u', u'' in the following way

$$u' = \sum_{i=1}^{n'} z_i'^1 \otimes ... \otimes z_i'^m \otimes t_i', \qquad u'' = \sum_{i=1}^{n''} z_i''^1 \otimes ... \otimes z_i''^m \otimes t_i'',$$

with

$$\begin{aligned} z_i'^1 &= \frac{(d_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p}}}{\delta_{p\sigma}((x_i'^j)_{i=1}^{n'})} x_i'^1, \quad z_i'^j = x_i'^j, j = 2, ..., m, i = 1, ..., n', \\ t_i' &= \frac{\delta_{p\sigma}((x_i'^j)_{i=1}^{n'})}{(d_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p}}} y_i', i = 1, ..., n', \\ z_i''^1 &= \frac{(d_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p}}}{\delta_{p\sigma}((x_i''^j)_{i=1}^{n''})} x_i''^1, \quad z_i''^j = x_i''^j, j = 2, ..., m, i = 1, ..., n'', \\ t_i'' &= \frac{\delta_{p\sigma}((x_i''^j)_{i=1}^{n''})}{(d_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p}}} y_i'', i = 1, ..., n''. \end{aligned}$$

It follows that

$$\begin{aligned} &\delta_{p\sigma}((z_i'^j)_{i=1}^{n'}) = (d_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p}}, j = 1, ..., m \quad \text{and} \quad \left\| (t_i')_{i=1}^{n'} \right\|_r \leq (d_{p,\sigma}(u') + \varepsilon)^{\frac{1}{r}}, \\ &\delta_{p\sigma}((z_i''^j)_{i=1}^{n''}) = (d_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p}}, j = 1, ..., m \quad \text{and} \quad \left\| (t_i'')_{i=1}^{n''} \right\|_r \leq (d_{p,\sigma}(u'') + \varepsilon)^{\frac{1}{r}}. \end{aligned}$$

Thus

$$\begin{aligned} \delta_{p\sigma}((z_i'^j)_{i=1}^{n'}). \left\| (t_i')_{i=1}^{n'} \right\|_r &\leq d_{p,\sigma}(u') + d_{p,\sigma}(u'') + 2\varepsilon, \\ \delta_{p\sigma}((z_i''^j)_{i=1}^{n''}). \left\| (t_i'')_{i=1}^{n''} \right\|_r &\leq d_{p,\sigma}(u') + d_{p,\sigma}(u'') + 2\varepsilon. \end{aligned}$$

The two last inequalities imply that $d_{p,\sigma}(u' + u'') \leq d_{p,\sigma}(u') + d_{p,\sigma}(u'') + 2\varepsilon$, hence the triangular inequality is proved for $d_{p,\sigma}$. It is easy to see that $d_{p,\sigma}(\lambda u) = |\lambda| \, d_{p,\sigma}(u)$ for all $u \in X_1 \otimes ... \otimes X_m \otimes Y$ and $\lambda \in \mathbb{K}$. Now, let $u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i \in X_1 \otimes ... \otimes X_m \otimes Y$, $\psi \in B_{Y^*}$ and $\phi_j \in B_{X_j^*}, j = 1, ..., m$. By Hölder's inequality we get

$$\begin{aligned} & \left| \sum_{i=1}^{n} \phi_1(x_i^1) ... \phi_m(x_i^m) \psi(y_i) \right| \\ \leq \;& \left(\sum_{i=1}^{n} \left| \phi_1(x_i^1) ... \phi_m(x_i^m) \right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \left\| (y_i)_{i=1}^{n} \right\|_r \\ \leq \;& \sup_{\phi \in B_{\mathcal{L}(X_1, ..., X_m)}} \left(\sum_{i=1}^{n} \left| \phi(x_i^1, ..., x_i^m) \right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \left\| (y_i)_{i=1}^{n} \right\|_r \\ \leq \;& \delta_{p\sigma}((x_i^j)_{i=1}^{n}) \left\| (y_i)_{i=1}^{n} \right\|_r . \end{aligned}$$

Then $\varepsilon(u) \leq \delta_{p\sigma}((x_i^j)_{i=1}^{n}) \left\| (y_i)_{i=1}^{n} \right\|_r$. Since this holds for every representation of u, we obtain $\varepsilon(u) \leq d_{p,\sigma}(u)$. Thus $d_{p,\sigma}(u) = 0$ implies $u = 0$. Hence $d_{p,\sigma}$ is a norm on $X_1 \otimes ... \otimes X_m \otimes Y$. It is clear that $d_{p,\sigma}(x^1 \otimes ... \otimes x^m \otimes y) \leq$

$\left\|x^1\right\| ... \|x^m\| \|y\|$ for every $x^j \in X_j, j = 1, ..., m$ and $y \in Y$. Let $\phi_j \in X_j^*$ with $\phi_j \neq 0, j = 1, ..., m$, let $\psi \in Y^*$ and let $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$. Then applying Hölder's inequality yields

$$\begin{aligned}
&|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(u)| \\
&= \left|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right| \\
&\leq \left(\sum_{i=1}^n \left|\phi_1(x_i^1)...\phi_m(x_i^m)\right|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \|(\psi(y_i))_{i=1}^n\|_r \\
&\leq \|\phi_1\| ... \|\phi_m\| \|\psi\| \sup_{\phi \in B_{\mathcal{L}(X_1,...,X_m)}} \left(\sum_{i=1}^n \left|\phi(x_i^1, ..., x_i^m)\right|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \|(y_i)_{i=1}^n\|_r \\
&\leq \|\phi_1\| ... \|\phi_m\| \|\psi\| \delta_{p\sigma}((x_i^j)_{i=1}^n) \|(y_i)_{i=1}^n\|_r .
\end{aligned}$$

It follow that $|\phi_1 \otimes ... \otimes \phi_m \otimes \psi(u)| \leq \|\phi_1\| ... \|\phi_m\| \|\psi\| d_{p,\sigma}(u)$. Therefore $\phi_1 \otimes ... \otimes \phi_m \otimes \psi$ is bounded and satisfies $|\phi_1 \otimes ... \otimes \phi_m \otimes \psi| \leq \|\phi_1\| ... \|\phi_m\| \|\psi\|$ and we have shown that $d_{p,\sigma}$ is a reasonable crossnorm. It only remains to show that $d_{p,\sigma} \leq \pi$. For every representation $\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$, of $u \in X_1 \otimes ... \otimes X_m \otimes Y$ we have

$$\begin{aligned}
d_{p,\sigma}(u) &\leq \delta_{p\sigma}((x_i^j)_{i=1}^n) \|(y_i)_{i=1}^n\|_r \\
&\leq \left(\sum_{i=1}^n \prod_{j=1}^m \left\|x_i^j\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \left(\sum_{i=1}^n \|y_i\|^r\right)^{\frac{1}{r}}
\end{aligned}$$

In the representation of u, replacing x_i^j by $\frac{\left(\prod_{k=1}^m \|x_i^k\| \|y_i\|\right)^{\frac{1}{q_j}}}{\|x_i^j\|} x_i^j$ and y_i by $\frac{\left(\prod_{k=1}^m \|x_i^k\| \|y_i\|\right)^{\frac{1}{r}}}{\|y_i\|} y_i$ with $q_1, ..., q_m > 1$ such that $\frac{1}{q_1} + ... + \frac{1}{q_m} = \frac{1-\sigma}{p}$, by a simple calculation, we obtain $d_{p,\sigma}(u) \leq \sum_{i=1}^n \prod_{k=1}^m \left\|x_i^k\right\| \|y_i\|$. Taking the infimum over all representation of u, we find $d_{p,\sigma}(u) \leq \pi(u)$. □

In what follows, we consider the tensor product of linear operators in connection with the reasonable crossnorm $d_{p,\sigma}$. We show that the reasonable crossnorm $d_{p,\sigma}$ is actually a tensor norm [11, Page 127].

Proposition 2.2. *Let X_j, Y_j, X, Y be Banach spaces, $p \geq 1$, $0 \leq \sigma < 1$, $T \in \mathcal{L}(X, Y)$ and $T_j \in \mathcal{L}(X_j, Y_j)$, $(j = 1, ..., m)$. Then there is a unique continuous linear operator*

$$T_1 \otimes_{d_{p,\sigma}} ... \otimes_{d_{p,\sigma}} T_m \otimes_{d_{p,\sigma}} T : \left(X_1 \widehat{\otimes} ... \widehat{\otimes} X_m \widehat{\otimes} X, d_{p,\sigma}\right) \longrightarrow \left(Y_1 \widehat{\otimes} ... \widehat{\otimes} Y_m \widehat{\otimes} Y, d_{p,\sigma}\right),$$

such that

$$T_1 \otimes_{d_{p,\sigma}} ... \otimes_{d_{p,\sigma}} T_m \otimes_{d_{p,\sigma}} T(x^1 \otimes ... \otimes x^m \otimes x) = (T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx),$$

for every $x^j \in X_j, (j = 1, ..., m)$ and $x \in X$. Moreover

$$\left\|T_1 \otimes_{d_{p,\sigma}} ... \otimes_{d_{p,\sigma}} T_m \otimes_{d_{p,\sigma}} T\right\| = \|T_1 \otimes ... \otimes T_m \otimes T\| = \|T\| \prod_{j=1}^m \|T_j\| .$$

Proof. By [11, Page 7] there is a unique linear operator

$$T_1 \otimes ... \otimes T_m \otimes T : (X_1 \otimes ... \otimes X_m \otimes X) \longrightarrow (Y_1 \otimes ... \otimes Y_m \otimes Y),$$

such that $T_1 \otimes ... \otimes T_m \otimes T(x^1 \otimes ... \otimes x^m \otimes x) = (T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx)$ for every $x^j \in X_j, j = 1, ..., m$ and $x \in X$. We may suppose $T_j \neq 0$, $j = 1, ..., m$ and $T \neq 0$. Let $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes x_i \in X_1 \otimes ... \otimes X_m \otimes X$, hence the sum $\sum_{i=1}^n \left(T_1 x_i^1\right) \otimes ... \otimes (T_m x_i^m) \otimes (T x_i)$ is a representation of $T_1 \otimes ... \otimes T_m \otimes T(u)$ in $Y_1 \otimes ... \otimes Y_m \otimes Y$. Then, for $p \geq 1$, $0 \leq \sigma < 1$ and $r \geq 1$ with $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we have

$$\begin{aligned}
&d_{p,\sigma}\left(T_1 \otimes ... \otimes T_m \otimes T(u)\right) \\
\leq\ & \delta_{p\sigma}((T_j x_i^j)_{i=1}^n) \|(T x_i)_{i=1}^n\|_r \\
\leq\ & \|T\| \prod_{j=1}^m \|T_j\| \delta_{p\sigma}((x_i^j)_{i=1}^n) \|(x_i)_{i=1}^n\|_r .
\end{aligned}$$

Since this holds for every representation of u, we get

$$d_{p,\sigma}\left(T_1 \otimes \ldots \otimes T_m \otimes T(u)\right) \leq \|T\| \prod_{j=1}^{m} \|T_j\| \, d_{p,\sigma}(u).$$

This means that $T_1 \otimes \ldots \otimes T_m \otimes T$ is bounded for the crossnorms on $d_{p,\sigma}$ and

$$\|T_1 \otimes \ldots \otimes T_m \otimes T\| \leq \|T\| \prod_{j=1}^{m} \|T_j\|.$$

On the other hand, as $d_{p,\sigma}$ is an reasonable crossnorm, we get that

$$\begin{aligned} \|Tx\| \prod_{j=1}^{m} \left\|T_j x^j\right\| &= d_{p,\sigma}\left((T_1 x^1) \otimes \ldots \otimes (T_m x^m) \otimes (Tx)\right) \\ &\leq \|T_1 \otimes \ldots \otimes T_m \otimes T\| \, d_{p,\sigma}\left(x^1 \otimes \ldots \otimes x^m \otimes x\right) \\ &= \|T_1 \otimes \ldots \otimes T_m \otimes T\| \, \|x\| \prod_{j=1}^{m} \left\|x^j\right\|. \end{aligned}$$

Thus $\|T_1 \otimes \ldots \otimes T_m \otimes T\| \geq \|T\| \prod_{j=1}^{m} \|T_j\|$ and therefore

$$\|T_1 \otimes \ldots \otimes T_m \otimes T\| = \|T\| \prod_{j=1}^{m} \|T_j\|.$$

Now, taking the unique continuous extension of the operator $T_1 \otimes \ldots \otimes T_m \otimes T$ to the completions of $X_1 \otimes \ldots \otimes X_m \otimes X$ and $Y_1 \otimes \ldots \otimes Y_m \otimes Y$, which we denote by $T_1 \otimes_{d_{p,\sigma}} \ldots \otimes_{d_{p,\sigma}} T_m \otimes_{d_{p,\sigma}} T$, we obtain a unique linear operator from $\left(X_1 \widehat{\otimes}_{d_{p,\sigma}} \ldots \widehat{\otimes}_{d_{p,\sigma}} X_m \widehat{\otimes}_{d_{p,\sigma}} X, d_{p,\sigma}\right)$ into $\left(Y_1 \widehat{\otimes}_{d_{p,\sigma}} \ldots \widehat{\otimes}_{d_{p,\sigma}} Y_m \widehat{\otimes}_{d_{p,\sigma}} Y, d_{p,\sigma}\right)$ with the norm

$$\left\|T_1 \otimes_{d_{p,\sigma}} \ldots \otimes_{d_{p,\sigma}} T_m \otimes_{d_{p,\sigma}} T\right\| = \|T\| \prod_{j=1}^{m} \|T_j\|.$$

□

Follows the idea of [9, Theorem 3.7] we prove the following result

Theorem 2.3. *The space* $\left(\mathcal{L}_p^{s,\sigma}\left(X_1, \ldots, X_m; Y^*\right), \|\cdot\|_{\mathcal{L}_p^{s,\sigma}}\right)$ *is isometrically isomorphic to* $\left(X_1 \otimes \ldots \otimes X_m \otimes Y, d_{p,\sigma}\right)^*$ *through the mapping* Ψ *defined by*

$$\Psi(T)(x^1 \otimes \ldots \otimes x^m \otimes y) = T(x^1, \ldots, x^m)(y),$$

for every $T \in \mathcal{L}_p^{s,\sigma}\left(X_1, \ldots, X_m; Y^*\right)$, $x^j \in X_j$, $j = 1, \ldots, m$, *and* $y \in Y$.

Proof. It is easy to see that the correspondence Ψ defined as above is linear. It remains to show the surjectivity and that $\|\Psi(T)\|_{(X_1 \otimes \ldots \otimes X_m \otimes Y, \beta_{p,\sigma})^*} = \|T\|_{\mathcal{L}_p^{s,\sigma}}$ for all T in $\mathcal{L}_p^{s,\sigma}\left(X_1, \ldots, X_m; Y^*\right)$. Let $\phi \in (X_1 \otimes \ldots \otimes X_m \otimes Y, d_{p,\sigma})^*$ and we take $T \in \mathcal{L}_p^{s,\sigma}\left(X_1, \ldots, X_m; Y^*\right)$ defined by $T(x^1, \ldots, x^m)(y) = \phi(x^1 \otimes \ldots \otimes x^m \otimes y)$. Let $(x_i^1, \ldots, x_i^m)_{i=1}^n \subset X_1 \times \ldots \times X_m$. For each $\varepsilon > 0$, choose $(y_i)_{i=1}^n \subset Y$, $\|y_i\| = 1$, $i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} \left\|T(x_i^1, \ldots, x_i^m)\right\|^{\frac{p}{1-\sigma}} \leq \varepsilon + \sum_{i=1}^{n} \left|T(x_i^1, \ldots, x_i^m)(y_i)\right|^{\frac{p}{1-\sigma}}. \tag{2.1}$$

Now, for $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ we have

$$\begin{aligned} &\left|\sum_{i=1}^{n} \lambda_i T(x_i^1, \ldots, x_i^m)(y_i)\right| \\ = &\left|\phi\left(\sum_{i=1}^{n} \lambda_i x_i^1 \otimes \ldots \otimes x_i^m \otimes y_i\right)\right| \\ \leq &\|\phi\| \, d_{p,\sigma}\left(\sum_{i=1}^{n} x_i^1 \otimes \ldots \otimes x_i^m \otimes (\lambda_i y_i)\right) \\ \leq &\|\phi\| \, \delta_{p\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda_i)_{i=1}^n\|_r. \end{aligned}$$

Taking the supermum over all $(\lambda_i)_{i=1}^n \subset \mathbb{K}$ such that $\|(\lambda_i)_{i=1}^n\|_r \leq 1$, we obtain

$$\left\| \left(T(x_i^1, ..., x_i^m)(y_i) \right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \leq \|\phi\| .\delta_{p\sigma}((x_i^j)_{i=1}^n).$$

Since ε is arbitrary, the latter inequality together (2.1) imply that

$$\left(\sum_{i=1}^n \left\| T(x_i^1, ..., x_i^m) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq \|\phi\| \, \delta_{p\sigma}((x_i^j)_{i=1}^n).$$

Showing that $T \in \mathcal{L}_p^{s,\sigma}(X_1, ..., X_m; Y^*)$ and $\|T\|_{\mathcal{L}_p^{s,\sigma}} \leq \|\phi\|$. Conversely, take $T \in \mathcal{L}_p^{s,\sigma}(X_1, ..., X_m; Y^*)$ and define a linear functional ϕ_T on $X_1 \otimes ... \otimes X_m \otimes Y$ by $\phi_T(u) = \sum_{i=1}^n T(x_i^1, ..., x_i^m)(y_i)$, where $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$, with $m \in \mathbb{N}$, $x_i^j \in X_j$, $y_i \in Y$, $i = 1, ..., n$, $j = 1, ..., m$. An application of Hölder's inequality reveals that,

$$\begin{aligned} |\phi_T(u)| &\leq \sum_{i=1}^n \left| T(x_i^1, ..., x_i^m)(y_i) \right| \\ &\leq \left\| \left(T(x_i^1, ..., x_i^m) \right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \|(y_i)_{i=1}^n\|_r \\ &\leq \|T\|_{\mathcal{L}_p^{s,\sigma}} \delta_{p\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_r . \end{aligned}$$

Thus $|\phi_T(u)| \leq \|T\|_{\mathcal{L}_p^{s,\sigma}} d_{p,\sigma}(u)$. This shows that $\phi_T \in (X_1 \otimes ... \otimes X_m \otimes Y, d_{p,\sigma})^*$ with $\|\phi_T\| \leq \|T\|_{\mathcal{L}_p^{s,\sigma}}$, and the proof concludes. □

ACKNOWLEDGMENT

The author wishes to thank the referees and editors for their valuable comments and suggestions which led to improvements in the document.

DECLARATION

The author declares no conflict of interest.

REFERENCES

[1] Achour, D., Dahia, E., Rueda, P., & Sánchez-Pérez, E. A. (2016). Domination spaces and factorization of linear and multilinear summing operators. *Quaestiones Mathematicae*, 39(8), 1071-1092.

[2] Achour, D., Dahia, E., Rueda, P., & Sánchez Pérez, E. A. (2014). Factorization of strongly (p, σ)-continuous multilinear operators. *Linear and Multilinear Algebra*, 62(12), 1649-1670.

[3] Achour, D., Dahia, E., Rueda, P., & Sánchez-Pérez, E. A. (2013). Factorization of absolutely continuous polynomials. *Journal of Mathematical Analysis and Applications*, 405(1), 259-270.

[4] Dahia, E. (2023). Tensorial representation of p-regular multilinear operators between Banach lattices. *Positivity*, 27(5), 65.

[5] Dahia, E. (2014). *On the tensorial representation of multi-linear ideals* (Doctoral dissertation, Ph. D. thesis, University of Mohamed Boudiaf, M'sila (Algeria)).

[6] Dahia, E., Achour, D., & Sánchez-Pérez, E. A. (2013). Absolutely continuous multilinear operators. *Journal of Mathematical Analysis and Applications*, 397(1), 205-224.

[7] Dimant, V. (2003). Strongly p-summing multilinear operators. *Journal of mathematical analysis and applications*, 278(1), 182-193.

[8] Molina, J. L., & Sánchez-Pérez, E. A. (1993). Ideales de operadores absolutamente continuos. Rev. *Real Acad. Ciencias Exactas, Fisicas y Naturales, Madrid*, 87, 349-378.

[9] Matos, M. C. (1993). On multilinear mappings of nuclear type. *Rev. Mat. Univ. Complut. Madrid*, 6(1), 61-81.

[10] Matter, U. (1987). Absolutely continuous operators and super-reflexivity. *Mathematische Nachrichten*, 130(1), 193-216..

[11] Ryan, R. A. (2002). *Introduction to tensor products of Banach spaces* (Vol. 73). London: Springer.