IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

Moderate Deviations Principle and Central Limit Theorem for Stochastic Cahn-Hilliard Equation in Hölder Norm.

Ratsarasaina R. M.[1] and Rabeherimanana T.J.[2]

ABSTRACT: We consider a stochastic Cahn-Hilliard partial differential equation driven by a space-time white noise. In this paper, we prove a Central Limit Theorem (CLT) and a Moderate Deviation Principle (MDP) for a perturbed stochastic Cahn-Hilliard equation in Hölder norm. The techniques are based on Freidlin-Wentzell's Large Deviations Principle. The exponential estimates in the space of Hölder continuous functions and the Garsia-Rodemich-Rumsey's lemma plays an important role, an another approach than the Li.R. and Wang.X. Finally, we estabish the CLT and MDP for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients.

Keywords: Large Deviations Principle, Moderate Deviations Principle, Central Limit Theorem, Hölder space, Stochastic Cahn-Hilliard equation, Green's function, Freidlin-Wentzell's method.

MSC: 60H15, 60F05, 35B40, 35Q62

1 Introduction and preliminaries.

The Cahn-Hilliard equation was developed in 1958 to model the phase separation process of a binary mixture (Cahn J.W. and Hilliard J.E. [3,4]). This approach has been extended to many other branches of science as dissimilar as polymer systems, population growth, image processing, spinodal decomposition, among others.

Consider the process $\{X^\varepsilon(t,x)\}_{\varepsilon>0}$ solution of stochastic Cahn-Hilliard with multipicative space time white noise, indexed by $\varepsilon > 0$, given by

$$\begin{cases} \partial_t X^\varepsilon(t,x) = -\Delta(\Delta X^\varepsilon(t,x) - f(X^\varepsilon(t,x))) + \sqrt{\varepsilon}\sigma(X^\varepsilon(t,x))\dot{W}(t,x), \\ \quad \text{in } \ (t,x) \in [0,T] \times D, \\ \\ X^\varepsilon(0,x) = X_0(x), \\ \\ \frac{\partial X^\varepsilon(t,x)}{\partial \mu} = \frac{\partial \Delta X^\varepsilon(t,x)}{\partial \mu} = 0, \quad \text{on} \ \ (t,x) \in [0,T] \times \partial D. \end{cases} \tag{1.1}$$

where $T > 0$, $D = [0,\pi]^3$, $\Delta X^\varepsilon(t,x)$ denotes the Laplacian of $X^\varepsilon(t,x)$ in the x-variable, μ is the outward normal vector, f is a polynomial of degree 3 with positive dominant coefficient such as $f = F'$ where

- [1] *Ratsarasaina R. M., Faculty of Sciences Technology, Departement of Mathematics and Informatics, University of Antananarivo, B.P.906, Ankatso, 101, Antananarivo, Madagascar.*
  E-mail: ratsarasainaralphmartial@gmail.com

- [2] *Rabeherimanana T.J., corresponding author, Faculty of Sciences Technology, Departement of Mathematics and Informatics, University of Antananarivo, B.P.906, Ankatso, 101, Antananarivo, Madagascar.*
  E-mail: rabeherimanana.toussaint@gmx.fr

Communicated Editor: Chala Adel
Manuscript received Dec 07, 2023; revised Fb 09, 2024; accepted Feb 16, 2024; published May 05, 2024.

$F(u) = (1-u^2)^2$, W is a space-time of a Brownian sheet defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is the formal derivative of a Brownian sheet W defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients f, σ are uniform Lipschitz with respect to x, with at most linear growth. More precisely, we suppose that there exists two constants K_f and K_σ such that $\forall x, y \in \mathbb{R}$,

$$\begin{cases} |f(x) - f(y)| \leq K_f |x-y| \\ \\ |\sigma(x) - \sigma(y)| \leq K_\sigma |x-y| \end{cases} \tag{1.2}$$

and that there exists a constant $K > 0$ such that :

$$\sup\{|f(x)|+|\sigma(x)|\} \leq K(1+|x|). \tag{1.3}$$

Let X^0 be the solution of the determinic Cahn-Hilliard equation

$$\partial_t X^0(t,x) = -\Delta(\Delta X^0(t,x) - f(X^0(t,x)))$$

with initial condition $X^0(0,x) = X_0(x)$. We expect that $||X^\varepsilon - X^0||_\alpha \to 0$ in probability as $\varepsilon \to 0^+$ where $||.||_\alpha$ is the Hölder norm (see (2.1)). The LDP, CLT and MDP for stochastic Cahn-Hilliard equation are not new. For example, Boulanba.L. and Mellouk.M. [2] studied the LDP for the mild solution of Stochastic Cahn-Hilliard equation (1.1). Li.R. and Wang.X. [8] studied the CLT and MDP for stochastic perturbed Cahn-Hilliard equation using the weak convergence approach.

However, we study its CLT and MDP for stochastic Cahn-Hilliard equation in the context of Hölder norm using another method. It means, we study the process

$$\eta^\varepsilon(t,x) = \left(\frac{X^\varepsilon - X^0}{\sqrt{\varepsilon}}\right)(t,x) \tag{1.4}$$

and

$$\theta^\varepsilon(t,x) = \left(\frac{X^\varepsilon - X^0}{\sqrt{\varepsilon} h(\varepsilon)}\right)(t,x) \tag{1.5}$$

in order to get a CLT and a MDP respectively.
The techniques are based on the exponential estimates in the space of Hölder continuous functions. The Garsia-Rodemich-Rumsey's lemma plays a very important role.
The paper is organized as follows : in the section one, we prove that $\eta^\varepsilon(t,x)$ defined by (1.4) converges in probability to $\eta^0(t,x)$. More precisely we purpose to prove that $\lim_{\varepsilon\to 0} \mathbb{E}||\eta^\varepsilon - \eta^0||_\alpha^r = 0$. In the section two, we study the LDP for (1.4) as $\varepsilon \to 0$ for $1 < h(\varepsilon) < \frac{1}{\sqrt{\varepsilon}}$, that is to say , the process $\theta^\varepsilon(t,x)$ defined by (1.5) obeys a LDP on $\mathcal{C}^\alpha([0,1] \times D)$ with speed $h^2(\varepsilon)$ and with rate function $\widetilde{I}(.)$ defined later. In section three, we prove the main results. Finally the example for CLT and MDP for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients be given in section four.

2 Main results

Let $\mathbb{H}$ denote the Cameron-Martin space associated with the Brownian sheet $\{W(t,x),\ t \in [0,T],\ \ x \in D\}$, that is to say,

$$\mathbb{H} = \left\{ h(t) = \int_0^t \int_D |\dot{h}(t,x)|^2 dt dx : \dot{h} \in L^2\big([0,T] \times D\big) \right\}.$$

Let $\mathcal{E}_0$, $\mathcal{E}$ be polish space such that the initial condition $X_0(x)$ takes valued in a compact subspace of $\mathcal{E}_0$ and $\Theta^\varepsilon = \{\mathcal{G}^\varepsilon : \mathcal{E}_0 \times \mathcal{C}\big([0,T] \times D, \mathbb{R}\big) \to \mathcal{E},\ \ \varepsilon > 0\}$ a family of measurable maps valued in $\mathcal{E}$.
For $X_0 \in \mathcal{E}_0$, define $X^{\varepsilon, X_0} = \mathcal{G}^\varepsilon\big(X_0, \sqrt{\varepsilon} W\big)$ and for $n_0 \in \mathbb{N}$, consider the following $S^{n_0} = \{\Psi \in L^2([0,T] \times D) : \int_0^T \int_D \Psi^2(s,y) ds dy \leq n_0\}$ which is a compact metric space, equipped with the weak topology on $L^2([0,T] \times D)$.
We denote $||.||_\alpha$ the α-hölder norm such that

$$||F||_\alpha = ||F||_\infty + |F|_\alpha \tag{2.1}$$

where

$$||F||_\infty = \sup\{|F(s,x)| : (s,x)\in[0,T]\times D\},$$
$$|F|_\alpha = \sup\left\{\frac{|F(s_1,x_1)-F(s_2,x_2)|}{(|s_1-s_2|+|x_1-x_2|^2)^\alpha} : (s_1,x_1),(s_2,x_2)\in[0,T]\times D\right\}.$$

Let $\mathcal{C}^\alpha([0,T]\times D)$ the space of function $F:[0,T]\times D\longrightarrow\mathbb{R}$ such that $||F||_\alpha<+\infty$.
Schilder's theorem for the Brownian sheet asserts that the family
$\{\sqrt{\varepsilon}W(t,x):\ \varepsilon>0\}$ satisfies a LDP on $\mathcal{C}^\alpha([0,T]\times D)$, with the good rate function $I(.)$ defined by

$$I(h)=\begin{cases}\frac{1}{2}\int_0^T\int_D|\dot{h}(t,x)|^2dtdx & \text{for } h\in\mathbb{H}\\ +\infty & \text{otherwise,}\end{cases}$$

For $h\in\mathbb{H}$, let $X^h_{X_0}$ be the solution of the following deterministic partial differential equation

$$\partial_t X^h_{X_0}(t,x)=-\Delta(\Delta X^h_{X_0}(t,x)-f(X^h_{X_0}(t,x)))+\sigma(X^h_{X_0}(t,x))\dot{h}(t,x)$$

with initial condition

$$X^h_{X_0}(0,x)=X_0(x).$$

Theorem 1([2]): *Let σ be continuous on $\mathbb{R}$, f and σ satisfy conditions (1.2) and (1.3). Then, the law of $X^\varepsilon_{X_0}$ satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times D)$ with a good rate fuction $\widetilde{I}_{X_0}(.)$ defined by*

$$\widetilde{I}_{X_0}(\Phi)=\inf_{\left\{\dot{h}\in L^2([0,T]\times D)\,:\,\Phi=\mathcal{G}^0(X_0,I(h))\right\}}\left\{\frac{1}{2}\int_0^T\int_D\dot{h}^2(s,y)dsdy\right\}$$

and $+\infty$ otherwise.
See also for example [1,7].
In addition to (1.2) and (1.3), the coefficient f is differentiable with respect to x and the derivative f' is also uniformly Lipschitz. More precisely, there exists a constante C such that

$$|f'(x)-f'(y)|\le C|x-y| \tag{2.2}$$

for all $x,y\in\mathbb{R}$.
Combined with the uniform Lipschitz continuity of f, we have

$$|f'(x)|\le K_f. \tag{2.3}$$

2.1 Central Limit Theorem

In this section, our first main result is the following theorem :

Theorem 2: *Suppose that f, f' and σ satisfy conditions (1.2), (1.3), (2.2) and (2.3). Then for any $\alpha\in[0;\frac{1}{4})$, $r\ge1$, the process $\eta^\varepsilon(t,x)$ defined by (1.4) converges in L^r to the random process $\eta^0(t,x)$ as $\varepsilon\to0$ where $\eta^0(t,x)$ verifies the stochastic partial differential equation*

$$\partial_t\eta^0(t,x)=-\Delta(\Delta\eta^0(t,x)-f'(X^0(t,x))\eta^0(t,x))+\sigma(X^0(t,x))\dot{W}(t,x)$$

with initial condition $\eta^0(0,x)=0$.

Let $S(t)=e^{-A^2t}$ be the semi-group generated by the operator $A^2u:=\sum_{i=0}^\infty e^{-\mu_i^2t}u_iw_i$ where $u:=\sum_{i=0}^\infty u_iw_i$. Then the convolution semi-group (see Cardon-Weber.C [5]) is defined by $S(t)U(x)=\sum_{i=0}^\infty e^{-\mu_i^2t}w_i(x)w_i(y)$ for any $U(x)$ in $L^2(D)$, with the associated Green's function G_t such that $G_t(x,y)=\sum_{i=0}^\infty e^{-\mu_i^2t}w_i(x)w_i(y)$. **Lemma 1:** *There exists positive constants C, γ and γ' satisfying $\gamma<4-d$, $\gamma\le2$ and $\gamma'<1-\frac{d}{4}$ such that for all $y,z\in D$, $0\le s<t\le T$ and $0\le h\le t$, we have :*

1. $\int_0^t\int_D|G_r(x,y)-G_r(x,z)|^2dxdr\le C|y-z|^\gamma$,
2. $\int_0^t\int_D|G_{r+h}(x,y)-G_r(x,y)|^2dxdr\le C|h|^{\gamma'}$,

3. $\int_0^t \int_D |G_r(x,y)|^2 dxdr \le C|t-s|^\gamma$,
4. $\sup_{t\in[0,T]} \int_0^t \int_D |G_{t-u}(x,z) - G_{t-u}(y,z)|^p dudz \le C|x-y|^{3-p}\ ,\ p \in]\frac{3}{2}, 3[$,
5. $\sup_{x\in D} \int_0^s \int_D |G_{t-u}(x,z) - G_{s-u}(x,z)|^p dudz \le C|t-s|^{\frac{(3-p)}{2}}\ ,\ p \in]1,3[$,
6. $\sup_{x\in D} \int_t^s \int_D |G_u(x,z)|^p dudz \le C|t-s|^{\frac{(3-p)}{2}}\ ,\ p \in]1,3[$.

2.2 Moderate Deviations Principle

In this paper, our second main result is the MDP for the Stochastic Cahn-Hilliard equation. More precisely, we assume that the process $\{\theta^\varepsilon(t,x)\}_{\varepsilon>0}$ defined by (1.5) obeys a LDP on the space $\mathcal{C}^\alpha([0,1]\times D)$, with speed $h^2(\varepsilon)$ and rate function $\widetilde{I}_{X_0}(.)$.

Proposition 1: *If f and σ are Lipschitzian, then there exists $C(p,K,K_f,$ $T,X_0)$ depending on p, K, K_f, T, X_0 such that*

$$\mathbb{E}\big(||X^\varepsilon - X^0||_\infty\big)^p \le \varepsilon^{\frac{p}{2}} C(p,K,K_f,T,X_0) \longrightarrow 0 \ \text{ as } \ \varepsilon \to 0.$$

Theorem 3: *Let σ be continuous on $\mathbb{R}$ and f, f', σ satisfy the conditions (1.2), (1.3), (2.2) and (2.3).Then, the process $\{\theta^\varepsilon(t,x)\}_{\varepsilon>0}$ defined by (1.5) obeys a LDP on the space $\mathcal{C}^\alpha([0,1]\times D)$, with speed $h^2(\varepsilon)$ and rate function $\widetilde{I}_{X_0}(.)$ such that:*

$$\widetilde{I}_{X_0}(\phi) = \inf_{\{\dot{h}\in L^2([0,T]\times D)\,:\,\phi=\mathcal{G}^0(X_0,I(h))\}} \left\{\frac{1}{2}\int_0^T\int_D \dot{h}^2(s,y)dyds\right\}$$

and $+\infty$ otherwise.

3 Proof of main results

Proof of proposition 1: In Boulanba and Mellouk [2], we know that the stochastic Cahn-Hilliard equation has a solution $\{X^\varepsilon(t,x)\}_{\varepsilon>0}$ such that

$$\begin{aligned} X^\varepsilon(t,x) &= \int_D G_t(x,y)X_0(y)dy + \int_0^t\int_D \Delta G_{t-s}(x,y) f(X^\varepsilon(s,y))dsdy \\ &+ \sqrt{\varepsilon}\int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy). \end{aligned}$$

and that $||X^\varepsilon - X^0||_\alpha \to 0$ in probability as $\varepsilon \to 0^+$ where X^0 is the solution of

$$X^0(t,x) = \int_D G_t(x,y)X_0(y)dy + \int_0^t\int_D \Delta G_{t-s}(x,y) f(X^0(s,y))dsdy.$$

Then we have

$$\begin{aligned} \big(X^\varepsilon - X^0\big)(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\big[f(X^\varepsilon(s,y)) - f(X^0(s,y))\big]dsdy \\ &+ \sqrt{\varepsilon}\int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy). \end{aligned}$$

Using the inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$, we have

$$\begin{aligned} \big(||X^\varepsilon - X^0||_\infty\big)^p &\le 2^{p-1}\Bigg(\Bigg[\sup_{\substack{0\le s\le T\\ x\in D}}\Bigg|\int_0^t\int_D \Delta G_{t-s}(x,y)[f(X^\varepsilon(s,y)) \\ &- f(X^0(s,y))]dsdy\Bigg|\Bigg]^p \\ &+ \varepsilon^{\frac{p}{2}}\Bigg[\sup_{\substack{0\le s\le T\\ x\in D}}\Bigg|\int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy)\Bigg|\Bigg]^p\Bigg). \end{aligned}$$

Denote

$$\begin{aligned}\alpha_1^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)[f(X^\varepsilon(s,y))-f(X^0(s,y))]dsdy,\\ \alpha_2^\varepsilon(t,x) &= \int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

From (1.2), (1.3) and Hölder inequality, for $p>2$,

$$\mathbb{E}\big(||\alpha_1^\varepsilon||_\infty^T\big)^p \le K_f^p\Big(\sup_{\substack{0\le s\le T\\ x\in D}}\Big|\int_0^t\int_D \Delta G_t^q(x,y)dsdy\Big|\Big)^{\frac{p}{q}}\mathbb{E}\int_0^T |X_{X_0}^\varepsilon - X_{X_0}^0|^p dt$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$ such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, and for any $x,y\in D$, $t\in[0,T]$, by Burkholder's inequality for stochastic integrals against Brownian sheets (see Walsh.J.B. [9], page 315) and Hölder's inequality, we have

$$\begin{aligned}&\mathbb{E}\big(|\alpha_2^\varepsilon(t,x)-\alpha_2^\varepsilon(t,y)|^p\big)\\ &\le c_p\mathbb{E}\Big(\int_0^t\int_D |G_{t-u}(x,z)-G_{t-u}(y,z)|^2\sigma^2\big(X_{X_0}^\varepsilon(u,z)\big)dudz\Big)^{\frac{p}{2}}\\ &\le c_pK^p\Big(\int_0^t\int_D |G_{t-u}(x,z)-G_{t-u}(y,z)|^{2q'}dudz\Big)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\Big(\int_0^t\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\Big)^{\frac{p}{2p^{'}}}\\ &\le C(p,K,X_0)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}},\end{aligned}\tag{3.1}$$

where (1.3) and 4 in Lemma 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$ and $C(p,K,X_0)$ is independent of ε.

Similary, from 4, 5 and 6 in Lemma 1, for $0\le s\le t\le T$,

$$\begin{aligned}&\mathbb{E}\big(|\alpha_2^\varepsilon(t,y)-\alpha_2^\varepsilon(s,y)|^p\big)\\ &\le c_p\mathbb{E}\Big(\int_0^s\int_D |G_{t-u}(y,z)-G_{s-u}(y,z)|^2\sigma^2(X_{X_0}^\varepsilon(u,z))dudz\Big)^{\frac{p}{2}}\\ &\quad+c_p\mathbb{E}\Big(\int_s^t\int_D |G_{t-u}(y,z)|^2\sigma^2(X_{X_0}^\varepsilon(u,z))dudz\Big)^{\frac{p}{2}}\\ &\le c_pK^p\Big(\int_0^s\int_D |G_{t-u}(y,z)-G_{s-u}(y,z)|^{2q'}dudz\Big)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\Big(\int_0^s\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\Big)^{\frac{p}{2p^{'}}}\\ &\quad+c_pK^p\Big(\int_s^t\int_D |G_{t-u}(y,z)|^{2q'}dudz\Big)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\Big(\int_s^t\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\Big)^{\frac{p}{2p^{'}}}\\ &\le C(p,K,X_0)|x-y|^{\frac{(3-2q^{'})p}{4q^{'}}}\end{aligned}\tag{3.2}$$

Putting together (3.1) and (3.2), by Garsia-Rodemich-Rumsey (see Wang.R. and Zang.T. [10] or Corollary 1.2 in Walsh.J.B. [9]), there exist a random variable $K_{p,\varepsilon}(\omega)$ and a constant c such that

$$\begin{aligned}
&\mathbb{E}\big(|\alpha_2^{\varepsilon}(t,y)-\alpha_2^{\varepsilon}(s,y)|^p\big)\\
&\quad\le\quad K_{p,\varepsilon}(\omega)^p(|t-s|+|x-y|)^{\gamma}\bigg(\log\frac{c}{|t-s|+|x-y|}\bigg)^2 \qquad (3.3)
\end{aligned}$$

and

$$\sup_{\varepsilon}\mathbb{E}[K_{p,\varepsilon}^p]<+\infty.$$

choosing $s=0$ in (3.3), we obtain

$$\begin{aligned}
\mathbb{E}\big(\sup_{\substack{0\le s\le T\\ x\in D}}\big|\int_0^t\int_D G_{t-s}(x,y)\sigma(X^{\varepsilon}(s,y))W(ds,dy)\big|\big)^p &\le C(p,K,X_0)\sup_{\varepsilon}\mathbb{E}[K_{p,\varepsilon}^p]\\
&< +\infty. \qquad (3.4)
\end{aligned}$$

Putting (3.1), (3.2) and (3.3) together and using 6 in Lemma 1, there exists a constant $C(p,K,K_f,X_0)$ such that

$$\mathbb{E}(||X_t^{\varepsilon}-X_t^0||_{\infty}^T)^p \le C(p,K,K_f,X_0)\bigg(\mathbb{E}\int_0^t(||X_s^{\varepsilon}-X_s^0||_{\infty})^p ds+\varepsilon^{\frac{p}{2}}\bigg)$$

By Gronwall's inequality, we have

$$\mathbb{E}(||X_t^{\varepsilon}-X_t^0||_{\infty})^p \le \varepsilon^{\frac{p}{2}}C(p,K,K_f,X_0)e^{C(p,K,K_f,X_0)T}.$$

Putting $\varepsilon\to 0$, the proof is complete. □

Proof of Theorem 2 : The following Lemma is a consequence of Garsia-Rodemich-Rumsey's theorem.

Lemma 2: *Let $\widetilde{V}^{\varepsilon}(t,x)=\{V^{\varepsilon}(t,x)\ :\ (t,x)\in[0,T]\times D\}$ be a family of real-valued stochastic processes and let $p\in(0,\infty)$. Suppose that $\widetilde{V}^{\varepsilon}(t,x)$ satisfies the following assumptions :*

A-1°) *For any $(t,x)\in[0,T]\times D$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}|V^{\varepsilon}(t,x)|^p=0$$

A-2°) *There exists $\gamma>0$ such that for any (t,x), $(s,y)\in[0,T]\times D$*

$$\mathbb{E}|V^{\varepsilon}(t,x)-V^{\varepsilon}(s,y)|^p\le C(|t-s|+|x-y|^2)^{2+\gamma},$$

where C is a constant independent of ε.
In this case, for any $\alpha\in(0,\frac{\gamma}{k})$, $p\in[1,k)$,

$$\lim_{\varepsilon\to 0}\mathbb{E}||V^{\varepsilon}||_{\alpha}^p=0.$$

In this section, we prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}||X_t^{\varepsilon}-X_t^0||_{\alpha}^r=0.$$

Consider the process $\eta^{\varepsilon}(t,x)$ defined by (1.4) and

$$\begin{aligned}
X^{\varepsilon}(t,x) &= \int_D G_t(x,y)X_0(y)dy+\int_0^t\int_D\Delta G_{t-s}(x,y)f(X^{\varepsilon}(s,y))dsdy\\
&+ \sqrt{\varepsilon}\int_0^t\int_D G_{t-s}(x,y)\sigma(X^{\varepsilon}(s,y))W(ds,dy).
\end{aligned}$$

We know that $||X^{\varepsilon}-X^0||_{\alpha}\to 0$ in probability as $\varepsilon\to 0^+$ where X^0 is the solution of

$$X^0(t,x) = \int_D G_t(x,y)X_0(y)dy+\int_0^t\int_D\Delta G_{t-s}(x,y)f(X^0(s,y))dsdy.$$

In this case, we have

$$\begin{aligned}\eta^{\varepsilon}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^{\varepsilon}(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\Big)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^{\varepsilon}(s,y))W(ds,dy)\end{aligned}$$

then

$$\begin{aligned}\eta^{\varepsilon}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^{\varepsilon}(s,y))\eta^{\varepsilon}(s,y)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^{\varepsilon}(s,y))W(ds,dy).\end{aligned}$$

For $\varepsilon\to 0$, we have

$$\begin{aligned}\eta^{0}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^{0}(s,y))\eta^{0}(s,y)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^{0}(s,y))W(ds,dy).\end{aligned}$$

To this end, we verify (A-1), (A-2); for $V^{\varepsilon}=\eta^{\varepsilon}-\eta^0$, write

$$\begin{aligned}V^{\varepsilon}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^{\varepsilon}(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\\ &- f^{'}(X^0(s,y))\eta^0(s,y)\Big)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\big(\sigma(X^{\varepsilon}(s,y))-\sigma(X^0(s,y))\big)W(ds,dy).\end{aligned}$$

Let

$$\begin{aligned}k_1^{\varepsilon}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^{\varepsilon}(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\\ &\quad -f^{'}(X^0(s,y))\eta^{\varepsilon}(s,y)\Big)dsdy,\\ k_2^{\varepsilon}(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^0(s,y))\big(\eta^{\varepsilon}(s,y)-\eta^0(s,y)\big)dsdy,\\ k_3^{\varepsilon}(t,x) &= \int_0^t\int_D G_{t-s}(x,y)\big(\sigma(X^{\varepsilon}(s,y))-\sigma(X^0(s,y))\big)W(ds,dy).\end{aligned}$$

Now we shall divide the proof into the following two steps.

Step 1. Following the same calculation as the proof of (3.4) in proposition 1, we deduce that for $p>2$, $0\le t\le 1$

$$\begin{aligned}\mathbb{E}\big(||k_3^{\varepsilon}||_{\infty}^{t}\big) &\le C(p,K_{\sigma},T)\int_0^t \mathbb{E}\big(||X^{\varepsilon}-X^0||_{\infty}^{s}\big)^p ds\\ &\le \varepsilon^{\frac{p}{2}}C(p,K,K_{\sigma},T,X_0).\end{aligned}$$

By Taylor's formula, there exists a random field $\beta^{\varepsilon}(t,x)$ taking values in $(0,1)$ such that,

$$\begin{aligned}f(X^{\varepsilon}(s,y))-f(X^0(s,y)) &= f^{'}\big(X^0(s,y)+\beta^{\varepsilon}(t,x)(X^{\varepsilon}(s,y)-X^0(s,y))\big)\\ &\quad\times(X^{\varepsilon}(s,y)-X^0(s,y))\end{aligned}$$

Since $f^{'}$ is also Lipschitz continuous, we have

$$\big|f^{'}\big(X^0(s,y)+\beta^{\varepsilon}(t,x)(X^{\varepsilon}(s,y)-X^0(s,y))\big)-f^{'}\big(X^0(s,y)\big)\big|$$

$$\leq C\beta^{\varepsilon}(t,x)\big|X^{\varepsilon}(t,x)-X^{0}(t,x)\big|.$$

then

$$\begin{aligned}&\big|f'\big(X^0(s,y)+\beta^{\varepsilon}(t,x)(X^{\varepsilon}(s,y)-X^0(s,y))\big)-f'\big(X^0(s,y)\big)\big|\\&\leq C\big|X^{\varepsilon}(t,x)-X^{0}(t,x)\big|.\end{aligned}$$

Hence

$$\begin{aligned}\big|k_1^{\varepsilon}(t,x)\big| &\leq C\int_0^t\int_D \Delta G_{t-s}(x,y)\big|\big(X^{\varepsilon}(t,x)-X^{0}(t,x)\big)\eta^{\varepsilon}(s,y)\big|dsdy\\ &= \sqrt{\varepsilon}C\int_0^t\int_D \Delta G_{t-s}(x,y)\big(\eta^{\varepsilon}(s,y)\big)^2dsdy. \end{aligned} \tag{3.5}$$

By Hölder's inequality, for $p>2$

$$\begin{aligned}&\mathbb{E}\big(\big|k_1^{\varepsilon}\big|_{\infty}^{t}\big)^p\\ &\leq \varepsilon^{\frac{p}{2}}C^p\bigg(\sup_{0\leq s\leq T\quad,x\in D}\ \bigg|\int_0^t\int_D \Delta G_s^q(x,y)dsdy\bigg|\bigg)^{\frac{p}{q}}\times\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}||_{\infty}^{s}\big)^{2p}ds\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Using (2.2) and applying proposition 1, there exists a constant $C(p,K,K_f,C,K_\sigma,T,X_0)$ depending on p, K, K_f, C, K_σ, T, X_0 such that

$$\mathbb{E}\big(\big|k_1^{\varepsilon}(t,x)\big|\big)^p \leq \varepsilon^{\frac{1}{2}}C(p,K,K_f,C,K_\sigma,T,X_0) \tag{3.6}$$

Noticing that $|f'|\leq K_f$, by Hölder inequality, we deduce that for $p>2$

$$\begin{aligned}&\mathbb{E}\big(\big|k_2^{\varepsilon}(t,x)\big|\big)^p\\ \leq\ & K_f^p\Big(\sup_{\substack{0\leq s\leq T\\ x\in D}}\ \Big|\int_0^t\int_D \Delta G_s^q(x,y)dsdy\Big|\Big)^{\frac{p}{q}}\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p ds\end{aligned} \tag{3.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Putting (3.5), (3.6) and (3.7) together, we have

$$\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p\leq C(p,K,K_f,C,K_\sigma,T,X_0)\bigg(\varepsilon^{\frac{1}{2}}+\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p ds\bigg)$$

By Gronwall's inequality, we obtain

$$\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p\leq \varepsilon^{\frac{1}{2}}C(p,K,K_b,C,K_\sigma,T,X_0)\longrightarrow 0\ \text{ for }\ \varepsilon\to 0.$$

Step 2. We show that all the terms k_i^ε, $i=1,2,3$ satisfy the condition (A-2) in Lemma 2. For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$ such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, for all $x,y\in D$, $0\le t\le T$, by Burkholder's inequality and Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}\big|k_3^\varepsilon(t,x)-k_3^\varepsilon(t,y)\big|^p &\le C_p\mathbb{E}\bigg(\int_0^t\int_D|G_{t-u}(x,z)-G_{t-u}(y,z)|^2\\ &\quad\times(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z)))^2dudz\bigg)^{\frac{p}{2}}\\ &\le C_p\bigg(\int_0^t\int_D(|G_{t-u}(x,z)-G_{t-u}(y,z)|)^{2q^{'}}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\quad\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ &\le C(p,q^{'},K_\sigma,K,T)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}}\end{aligned}\tag{3.8}$$

where (1.3), 4 in Lemma 1 and Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$.
Similarly, in view of 5 , 6 in Lemma 1; it follows that for $0\le s\le t\le T$, we have

$$\begin{aligned}&\mathbb{E}\big|k_3^\varepsilon(t,y)-k_3^\varepsilon(s,y)\big|^p\\ \le\ & C_p\mathbb{E}\Big(\int_0^s\int_D|G_{t-u}(y,z)-G_{s-u}(y,z)|^2\big(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z))\big)^2dudz\Big)^{\frac{p}{2}}\\ +\ & C_p\mathbb{E}\Big(\int_s^t\int_D|G_{t-u}(y,z)|^2\big(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z))\big)^2dudz\Big)^{\frac{p}{2}}\\ \le\ & C_p\bigg(\int_0^t\int_D|G_{t-u}(y,z)-G_{s-u}(y,z)|^{2q^{'}}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ +\ & C_p\Big(\int_s^t\int_D|G_{t-u}(y,z)|^{2q^{'}}dudz\Big)^{\frac{p}{2q^{'}}}\\ &\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ \le\ & C(p,q^{'},K_\sigma,K,T)|t-s|^{\frac{(3-2q^{'})p}{4q^{'}}}\end{aligned}\tag{3.9}$$

where Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$, $C(p,q^{'},K_\sigma,K,T)$ is independent of ε.
Putting together (3.8) and (3.9), we have

$$\mathbb{E}\big|k_3^\varepsilon(t,x)-k_3^\varepsilon(s,y)\big|^p\le C(p,q^{'},K_\sigma,K,T)\big(|t-s|+|x-y|^2\big)^\gamma\tag{3.10}$$

Consequently, from 4, 6 in Lemma 1, proposition 1 and the result of step 1, we also have :

$$\mathbb{E}\big|k_i^\varepsilon(t,x)-k_i^\varepsilon(s,y)\big|^p\le C\big(|t-s|+|x-y|^2\big)^\gamma\ ,\ \ i=2,3.\tag{3.11}$$

Putting together (3.10) and (3.11), we obtain that there exists a constant C independent of ε satisfying that

$$\mathbb{E}\big|(\eta^\varepsilon(t,x)-\eta^0(t,x))-(\eta^\varepsilon(s,y)-\eta^0(s,y))|^p\le C\big(|t-s|+|x-y|^2\big)^\gamma$$

For any $\alpha\in(0,\frac{1}{4})$, $r\ge1$, choosing $p>2$, and $q^{'}\in(1,\frac{1}{4})$ such that $\alpha\in(0,\frac{\gamma}{p})$ and $r\in[1,p)$, Lemma 2 we have

$$\lim_{\varepsilon\to0}\mathbb{E}||\eta^\varepsilon-\eta||_\alpha^r=0.$$

The proof is complete . □

Proof of Theorem 3 : Recall the following lemma from Chenal.F and Millet.A [6].

Lemma 3: *Let $F : ([0,T]\times D)^2 \longrightarrow \mathbb{R}$, $\alpha_0 > 0$ and $C_F > 0$ be such that for any $(t,x),(s,y) \in [0,T]\times D$, set*

$$\int_0^T \int_D |F(t,x,u,z) - F(s,y,u,z)|^2 dudz \leq C(|t-s| + |x-y|^2)^{\alpha_0}. \tag{3.12}$$

Let $N : [0,T]\times D \longrightarrow \mathbb{R}$ be an almost surely continuous, $\mathcal{F}_t$−adapted such that $\sup\{|N(t,x)| : (t,x) \in [0,T]\times D\} \leq \rho$,a.s., and for $(t,x) \in [0,T]\times D$, set

$$\mathfrak{F}(t,x) = \int_0^T \int_D F(t,x,u,z)N(u,z)W(dudz)$$

Then for all $\alpha \in]0, \frac{\alpha_0}{2}[$, there exists a constant $C(\alpha,\alpha_0)$ such that for all $M \geq \rho C_F C(\alpha,\alpha_0)$

$$\mathbb{P}(||\mathfrak{F}||_\alpha \geq M) \leq (\sqrt{2}T^2 + 1)\exp\bigg(-\frac{M^2}{\rho^2 C_F C^2(\alpha,\alpha_0)}\bigg)$$

Proof of Theorem 3 : Now, we prove the MDP, that is to say, the process θ^ε defined by (1.5) obeys a LDP on $\mathcal{C}^\alpha([0,T]\times D)$, with the speed function $h^2(\varepsilon)$ and the rate function $\widetilde{I}(.)$. More precisely, to prove the LDP of $\frac{\eta^\varepsilon}{h(\varepsilon)}$, it is enough to show that $\frac{\eta^\varepsilon}{h(\varepsilon)}$ is $h^2(\varepsilon)$-exponentially equivalent to $\frac{\eta^0}{h(\varepsilon)}$,that is to say, for any $\delta > 0$, we have

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\bigg(\frac{||\eta^\varepsilon - \eta^0||_\alpha}{h(\varepsilon)} > \delta\bigg) = -\infty. \tag{3.13}$$

Since

$$||\eta^\varepsilon - \eta^0||_\alpha \leq (1 + (1+T)^\alpha)|\eta^\varepsilon - \eta^0|_\alpha^T$$

to prove (3.13), it is enough to prove that

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\bigg(\frac{|\eta^\varepsilon - \eta^0|_\alpha^T}{h(\varepsilon)} > \delta\bigg) = -\infty \quad , \qquad \forall\delta > 0.$$

Recall the decomposition in Proof of Theorem 2,

$$\eta^\varepsilon(t,x) - \eta^0(t,x) = k_1^\varepsilon(t,x) + k_2^\varepsilon(t,x) + k_3^\varepsilon(t,x).$$

For any q in $(\frac{3}{2},3)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $x,y \in D$, $0 \leq s \leq t \leq T$, by Hölder's inequality, 4 in Lemma 1 and (2.3), we have

$$\begin{aligned}
\big|k_2^\varepsilon(t,x) - k_2^\varepsilon(t,y)\big|^p &\leq K_f\bigg(\int_0^t \int_D |\Delta G_{t-u}(x,z) - \Delta G_{t-u}(y,z)|^q dudz\bigg)^{\frac{1}{q}} \\
&\quad \times\bigg(\int_0^t \int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p dudz\bigg)^{\frac{1}{p}} \\
&\leq K_f|x-y|^{\frac{3-q}{q}} \times \bigg(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\bigg)^{\frac{1}{p}}
\end{aligned} \tag{3.14}$$

Similarly, in view of 5 and 6 in Lemma 1, it follows that for $0 \le s \le t \le T$,

$$\begin{aligned} \left|k_2^\varepsilon(t,y) - k_2^\varepsilon(s,y)\right|^p &\le K_f\Big(\int_0^s\int_D |\Delta G_{t-u}(y,z) - \Delta G_{s-u}(y,z)|^q dudz\Big)^{\frac{1}{q}} \\ &\quad \times\Big(\int_0^s\int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p\Big)^{\frac{1}{p}} \\ &\quad +\Big(\int_s^t\int_D |\Delta G_{t-u}(y,z)|^q dudz\Big)^{\frac{1}{q}} \\ &\quad \times\Big(\int_0^t\int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p\Big)^{\frac{1}{p}} \\ &\le 2K_f|t-s|^{\frac{3-q}{2q}} \times \bigg(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\bigg)^{\frac{1}{p}} \end{aligned} \tag{3.15}$$

Putting together (3.14), (3.15), we have

$$\left|k_2^\varepsilon(t,y) - k_2^\varepsilon(s,y)\right|^p \le C(K_f)(|t-s| + |x-y|^2)^{\frac{3-q}{2q}} \times \bigg(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\bigg)^{\frac{1}{p}}.$$

Choosing $q \in (\frac{3}{2}, 3)$, such that $\alpha = (3-q)/2q$ and noticing that $||\eta^\varepsilon - \eta^0||_\infty^u \le (1+u)^\alpha|\eta^\varepsilon - \eta|_\alpha^u$, we obtain that

$$|k_2^\varepsilon|_\alpha^t \le C(K_f)\bigg(\int_0^t ((1+u)^\alpha|\eta^\varepsilon - \eta^0|_\alpha^u)^p du\bigg)^{\frac{1}{p}}$$

Thus, for $t \in [0,1]$, we have

$$(|\eta_t^\varepsilon - \eta_t^0|_\alpha^t)^p \le C(p,T,K_f)\bigg[\big(|k_1^\varepsilon(t)|_\alpha^t + |k_3^\varepsilon(t)|_\alpha^t\big)^p + \int_0^t (|\eta^\varepsilon - \eta^0|_\alpha^s)^p ds\bigg]$$

Applying Gronwall's Lemma, we have

$$(|\eta_t^\varepsilon - \eta_t^0|_\alpha^t)^p \le C(p,T,K_f)\bigg[\big(|k_1^\varepsilon(t)|_\alpha^t + |k_3^\varepsilon(t)|_\alpha^t\big)^p\bigg] e^{C(p,T,K_f)T} \tag{3.16}$$

By (3.15) and (3.16), its sufficient to prove that for any $\delta > 0$

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{|k_i^\varepsilon(t)|_\alpha^T}{h(\varepsilon)} > \delta\bigg) = -\infty \qquad i = 1,3.$$

Step 1. For any $\varepsilon > 0$, $\eta > 0$ we have

$$\begin{aligned} \mathbb{P}\big(|k_3^\varepsilon|_\alpha^T > h(\varepsilon)\delta\big) &\le \mathbb{P}\big(|k_3^\varepsilon|_\alpha^T > h(\varepsilon)\delta, |X^\varepsilon - X^0|_\infty^T < \eta\big) \\ &+ \mathbb{P}(|X^\varepsilon - X^0|_\infty^T \ge \eta) \end{aligned} \tag{3.17}$$

By 4 and 6 in Lemma 1, $G_{t-u}(x,z)1_{[u\le t]}$ satisfies (3.12)(see Lemma 3) for $\alpha_0 = \frac{1}{2}$.
Applying Lemma 3, we have

$$F(t,x,u,z) = G_{t-u}(x,z)1_{[u\le t]}, \alpha_0 = \frac{1}{2}, C_F = C, M = h(\varepsilon)\delta, \rho = \eta K_\sigma,$$

$$\widetilde{Y}(t,x) = \big(\sigma(X_{X_0}^\varepsilon(t,x)) - \sigma(X_{X_0}^0(t,x))\big)1_{||X^\varepsilon - X^0||_\infty^T > \eta}$$

, we obtain that for all ε sufficiently small such that $h(\varepsilon)\delta \ge \rho C C(\alpha, \frac{1}{2})$,

$$\begin{aligned} &\mathbb{P}\big(|k_3^\varepsilon(t)|_\alpha^T > h(\varepsilon)\delta, ||X^\varepsilon - X^0||_\infty^T < \eta\big) \\ &\quad \le (\sqrt{2}T^2 + 1)\exp\bigg(-\frac{h^2(\varepsilon)\delta^2}{\eta^2 K_\sigma^2 C C^2(\alpha, \frac{1}{2})}\bigg). \end{aligned} \tag{3.18}$$

Since $X^\varepsilon_{X_0}$ satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times D)$, see Theorem 1

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(||X^\varepsilon - X^0||^T_\infty \geq \eta) &\leq \limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(||X^\varepsilon - X^0||_\alpha \geq \eta)\\ &\leq -\inf\{I_{X_0}(f):\ ||f-X^0||_\alpha \geq \eta\}\end{aligned}$$

In this case, the good rate function $\mathcal{I} = \{I_{X_0}(f):\ ||f-X^0||_\alpha \geq \eta\}$ has compact level sets, the "$\inf\{I_{X_0}(f)\ :\ ||f-X^0||_\alpha \geq \eta\}$" is obtained at some function f_0. Because $I_{X_0}(f) = 0$ if and only if $f = X^0_{X_0}$, we conclude that

$$-\inf\{I_{X_0}(f)\ :\ ||f-X^0||_\alpha \geq \eta\} < 0.$$

For $h(\varepsilon)\to\infty$, $\sqrt{\varepsilon}h(\varepsilon)\to 0$, we have

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\big(||X^\varepsilon - X^0||^T_\infty \geq \eta\big) = -\infty. \tag{3.19}$$

Since $\eta > 0$ is arbitrary, putting together (3.17), (3.18) and (3.19), we obtain

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{||k^\varepsilon_3||_\alpha}{h(\varepsilon)} \geq \delta\bigg) = -\infty. \tag{3.20}$$

Step 2. For the first term $k^\varepsilon_1(t)$, let

$$k^\varepsilon_1(t,x) = \int_0^t\int_D \Delta G_{t-s}(x,y)\mathfrak{B}^\varepsilon(s,y)dsdy,$$

where

$$\mathfrak{B}^\varepsilon(s,y) = \bigg(\frac{f(X^\varepsilon(s,y)) - f(X^0(s,y))}{\sqrt{\varepsilon}} - f^{'}(X^0(s,y))\eta^\varepsilon(s,y)\bigg),$$

as stated in the proof of Theorem 2, we have

$$||\mathfrak{B}^\varepsilon||^T_\infty \leq C\frac{(||X^\varepsilon_{X_0} - X^0_{X_0}||^T_\infty)^2}{\sqrt{\varepsilon}}.$$

However, by Hölder's continuity of Green function G, it is easy to prove that, for any $\alpha\in(0,\frac{1}{4})$

$$|k^\varepsilon_2|^T_\alpha \leq C(\alpha,T)||\mathfrak{B}^\varepsilon||^T_\infty.$$

From the proof of proposition 1, we obtain that

$$||X^\varepsilon_{X_0} - X^0_{X_0}||^T_\infty \leq C(K_b,T)||\widetilde{k}^\varepsilon_2||^T_\infty$$

where

$$\widetilde{k}^\varepsilon_2(t,x) = \bigg(\varepsilon\int_0^t\int_D \Delta G_{t-s}(x,y)\sigma(X^\varepsilon_{X_0}(s,y))W(dsdy)\bigg)^{\frac{1}{2}}.$$

Applying lemma 3, we have

$$F(t,x,u,z) = G_{t-u}(x,z)1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, \rho = \sqrt{\varepsilon}K(1+||X^T_{X_0}||^T_\infty + \eta)$$
$$\widetilde{Z}(t,x) = \sqrt{\varepsilon}\sigma(X^\varepsilon_{X_0}(t,x))1_{[||X^\varepsilon_{X_0}||^T_\infty < ||X^0_{X_0}||^T_\infty + \eta]},$$

for any $\eta > 0$, we obtain that for all ε is sufficiently small such that
$M \geq \sqrt{\varepsilon}K(1+||X^T_{X_0}||^T_\infty + \eta)CC(\alpha,\frac{1}{2})$,

$$\begin{aligned}&\mathbb{P}(||\widetilde{k}^\varepsilon_2||^T_\infty \geq M, ||X^\varepsilon_{X_0}||^T_\infty < ||X^0_{X_0}||^T_\infty + \eta)\\ &\leq (\sqrt{2}T^2+1)\exp\bigg(-\frac{M^2}{\varepsilon K^2 CC^2(\alpha,\frac{1}{2})(1+||X^0_{X_0}||^T_\infty + \eta)^2}\bigg).\end{aligned}$$

For the same reason as (3.20), we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}||^T_\infty \geq ||X^0_{X_0}||^T_\infty+\eta) & \\ \leq & \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}-X^0_{X_0}||^T_\infty\geq\eta)\\ = & -\infty.\end{aligned}$$

For any $\eta>0$, by Bernstein's inequality and the continuity of σ, we have

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{|k^{\varepsilon}_1(t)|^T_\alpha}{h(\varepsilon)}\geq\delta\bigg)$$

$$\begin{aligned}\leq\ & \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\Bigg(\bigg(||\widetilde{k}^{\varepsilon}_2||^T_\infty\bigg)^2\geq\frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,T,K_f,C)}\Bigg)\\ \leq\ & \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\Bigg[\mathbb{P}\bigg(\big(||\widetilde{k}^{\varepsilon}_2(t)||^T_\infty\big)^2\geq\frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,T,K_f,C)},\\ & ||X^{\varepsilon}_{X_0}||<||X^0_{X_0}||^T_\infty+\eta\bigg)+\mathbb{P}(||X^{\varepsilon}_{X_0}||\geq||X^0_{X_0}||^T_\infty+\eta)\Bigg]\\ \leq\ & \Bigg(\limsup_{\varepsilon\to 0}\frac{-\delta}{\sqrt{\varepsilon}h(\varepsilon)C(\alpha,T,K_f,C)K^2CC^2(\alpha,\frac{1}{2})(1+||X_{X_0}||^T_\infty+\eta)^2}\Bigg)\\ \vee & \Bigg(\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}||\geq||X^0_{X_0}||^T_\infty+\eta)\Bigg)=-\infty.\end{aligned}$$ □

4 A FEW EXAMPLES

4.1 Example one. Central limit theorem for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients

Let $\mathcal{O}$ be an open connected set in $\mathbb{R}^3$ such that $\mathcal{O}=[0,\pi]^3$ and $\mathcal{C}^\alpha([0,1]\times\mathcal{O})$ denotes the set of α-Hölder continuous fonctions. Let $\{u^\varepsilon(t,x)\}_{\varepsilon>0}$ be the solution of stochastic Cahn-Hilliard equation indexed by $\varepsilon>0$, given by

$$\begin{cases}\partial_t u^\varepsilon(t,x)=-\Delta\big(\Delta u^\varepsilon(t,x)-4(u^\varepsilon(t,x))^3+4u^\varepsilon(t,x)\big)+\sqrt{\varepsilon}(1-u^\varepsilon(t,x))\dot{W},\\ \frac{\partial u^\varepsilon(t,x)}{\partial\nu}=\frac{\partial\Delta u^\varepsilon(t,x)}{\partial\nu}=0,\ \text{on}\quad (t,x)\in[0,T]\times\partial\mathcal{O}\\ u^\varepsilon(0,x)=u_0(x)\end{cases}\tag{4.1}$$

where the coefficients f and σ are bounded, uniformly Lipschitz and verify the condition (1.2) and (1.3) such that $K_f=16$ and $K_\sigma=1$. Consider the process $\beta^\varepsilon(t,x)$ such that

$$\beta^\varepsilon(t,x)=\bigg(\frac{u^\varepsilon-u^0}{\sqrt{\varepsilon}}\bigg)(t,x).\tag{4.2}$$

In this section, we establish the CLT for the stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients in Hölder norm $||.||_\alpha$ such that for all $u:[0,1]\times\mathcal{O}\longrightarrow\mathbb{R}$,

$$||u||_\alpha=\sup_{(s,x)\in[0,T]\times\mathcal{O}}|u(s,x)|+\sup_{\substack{(s_1,x_1)\in[0,T]\times\mathcal{O}\\(s_2,x_2)\in[0,T]\times\mathcal{O}}}\frac{|u(s_1,x_1)-u(s_2,x_2)|}{(|s_1-s_2|+|x_1-x_2|^2)^\alpha}.$$

Now, we obtain the main results similary to Theorem 2.

Theorem 5: *For any $\alpha\in[0,\frac{1}{4})$, $r\geq 1$, the process $\beta^\varepsilon(t,x)$ defined by (4.2) converges in L^r to the random process $\beta^0(t,x)$ as $\varepsilon\to 0$ where $\beta^0(t,x)$ verifies the stochastic partial differential equation*

$$\partial_t\beta^0(t,x)=-\Delta(\Delta\beta^0(t,x)-4\big(3(u^0(t,x))^2-1)\beta^0(t,x)\big)+(1-u^0(t,x))\dot{W}(t,x)$$

with initial condition $\eta^0(0,x)=0$.

Proof of Theorem 5 : Consider the process $\beta^\varepsilon(t,x)$ defined by (4.2) depending on $u^\varepsilon(t,x)$ and $u^0(t,x)$ such that

$$\begin{aligned}\beta^\varepsilon(t,x)& \\ &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\Big(\frac{(u^\varepsilon(s,y))^3-u^\varepsilon(s,y)-((u^0(s,y))^3-u^0(s,y))}{\sqrt{\varepsilon}}\Big)dsdy \\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

Using the equality $\forall a,b\neq 0, \frac{a^3-b^3}{a-b}=a^2+ab+b^2$, we obtain

$$\begin{aligned}\beta^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\big[(u^\varepsilon(s,y))^2+u^\varepsilon(s,y).u^0(s,y) \\ &+ (u^0(s,y))^2-1\big]\beta^\varepsilon(s,y)dsdy \\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^\varepsilon(s,y))W(ds,dy)\end{aligned}$$

For $\varepsilon\to 0$, we obtain

$$\begin{aligned}\beta^0(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\big(3(u^0(s,y))^2-1\big)\beta^0(s,y)dsdy \\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^0(s,y))W(ds,dy).\end{aligned}$$

Denote the process $\mathcal{R}^\varepsilon=\beta^\varepsilon-\beta^0$ such that

$$\mathcal{R}^\varepsilon = m_1^\varepsilon(t,x)+m_2^\varepsilon(t,x)+m_3^\varepsilon(t,x)$$

where

$$\begin{aligned}m_1^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\Big[\Big(\frac{(u^\varepsilon(s,y))^3-(u^0(s,y))^3}{\sqrt{\varepsilon}}\Big) \\ &-\Big(\frac{u^\varepsilon(s,y)-u^0(s,y)}{\sqrt{\varepsilon}}\Big)-\big(3(u^0(s,y))^2-1\big)\beta^\varepsilon(s,y)\Big]dsdy, \\ m_2^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\big(3(u^0(s,y))^2-1\big)\big(\beta^\varepsilon(s,y)-\beta^0(s,y)\big)dsdy, \\ m_3^\varepsilon(t,x) &= \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(u^0(s,y)-u^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

Step 1. For $p>2$ and $t\in[0,1]$, we obtain

$$\begin{aligned}\mathbb{E}\big(\big\|m_3^\varepsilon(t,x)\big\|_\infty^t\big) &\leq C(p,T)\int_0^t\mathbb{E}\big(\|u^\varepsilon-u^0\|_\infty^s\big)^p ds \\ &\leq \sqrt{\varepsilon}C(p,T,u_0).\end{aligned}$$

By Taylor's formula, there exists a random field $\gamma^\varepsilon(t,x)$ taking values in $[0,1]$ such that

$$\begin{aligned}f(u^\varepsilon(s,y))-f(u^0(s,y))& \\ &= f'\big(u^0(s,y)+\beta^\varepsilon(t,x)(u^\varepsilon(s,y)-u^0(s,y))\big)(u^\varepsilon(s,y)-u^0(s,y))\end{aligned}$$

For the first term $m_1^\varepsilon(t,x)$, we have

$$\big|m_1^\varepsilon(t,x)\big| \;\; \leq \;\; 4\sqrt{\varepsilon}C\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\big(\beta^\varepsilon(s,y)\big)^2 dsdy. \tag{4.3}$$

By Hölder's inequality, for $p>2$

$$\mathbb{E}\big(\big|m_1^\varepsilon(t,x)\big|_\infty^t\big)^p$$

$$\leq \;\; (\sqrt{\varepsilon})^p C^p\bigg(\sup_{0\leq s\leq T \;\;,\;\; x\in\mathcal{O}}\bigg|\int_0^t\int_{\mathcal{O}}\Delta G_s^q(x,y)dsdy\bigg|\bigg)^{\frac{p}{q}}\times\int_0^t\mathbb{E}\big(||\beta^\varepsilon||_\infty^s\big)^{2p}ds$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using (1.5) and applying proposition 1, there exists a constant $\aleph_{p,K,C}$ depending on p, K, C such that

$$\mathbb{E}\big|m_1^\varepsilon(t,x)\big|^p \;\; \leq \;\; \sqrt{\varepsilon}.\aleph_{p,K,C}. \tag{4.4}$$

Since $|f^{'}|\leq 16$,by Hölder inequality , we deduce that for $p>2$

$$\begin{aligned}\mathbb{E}|m_2^\varepsilon(t,x)|^p \;\; &\leq \;\; 2^{4p}\bigg(\sup_{0\leq s\leq T \;\;, x\in}\bigg|\int_0^t\int_{\mathcal{O}}\Delta G_s^q(x,y)dsdy\bigg|\bigg)^{\frac{p}{q}}\\ &\times\int_0^t\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p ds\end{aligned} \tag{4.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Putting (4.3),(4.4) and (4.5) together, we have

$$\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p \;\; \leq \;\; \aleph_{p,K,C}\big(\sqrt{\varepsilon}+\int_0^t\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p ds\big).$$

By Gronwall's inequality, we obtain

$$\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p \;\; \leq \;\; \sqrt{\varepsilon}\aleph_{p,K,C}\to 0 \;\text{ for }\; \varepsilon\to 0.$$

Step 2. We prove that the terms k_i^ε , $i=1,2,3$ satisfy the condition (A-2) in Lemma 2.
For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, for all $x,y\in\mathcal{O}$, $0\leq t\leq T$,by Burkholder's inequality and Hölder's inequality, we have

$$\mathbb{E}\big|m_3^\varepsilon(t,x)-m_3^\varepsilon(t,y)\big|^p \;\; \leq \;\; C(p,q^{'},K,T)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}} \tag{4.6}$$

where (1.3), 4 in Lemma 1 and Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$.
Similarly, in view of 5, 6 in Lemma 1; its follows that for $0\leq s\leq t\leq T$, we have

$$\mathbb{E}\big|m_3^\varepsilon(t,y)-m_3^\varepsilon(s,y)\big|^p \;\; \leq \;\; C(p,q^{'},K,T)|t-s|^{\frac{(3-2q^{'})p}{4q^{'}}} \tag{4.7}$$

where Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$, $C(p,q^{'},K,T)$ is independent of ε .
Putting together (4.6) and (4.7), we have

$$\mathbb{E}\big|m_3^\varepsilon(t,x)-m_3^\varepsilon(s,y)\big|^p\leq C(p,q^{'},K_\sigma,K,T)\big(|t-s|+|x-y|^2\big)^\gamma. \tag{4.8}$$

Consequently, from 4, 6 in Lemma 1, proposition 1 and the result of step 1, we also have :

$$\mathbb{E}\big|m_i^\varepsilon(t,x)-m_i^\varepsilon(s,y)\big|^p\leq C\big(|t-s|+|x-y|^2\big)^\gamma \;\;,\;\; i=2,3. \tag{4.9}$$

Putting together (4.8) and (4.9), we obtain that there exists a constant C independent of ε satisfying that

$$\mathbb{E}\big|(\beta^\varepsilon(t,x)-\beta^0(t,x))-(\beta^\varepsilon(s,y)-\beta^0(s,y))|^p\leq C\big(|t-s|+|x-y|^2\big)^\gamma.$$

For any $\alpha\in(0,\frac{1}{4})$, $r\geq 1$, choosing $p>2$, and $q^{'}\in(1,\frac{3}{2})$ such that $\alpha\in(0,\frac{\gamma}{p})$ and $r\in[1,p)$, Lemma 2 we have

$$\lim_{\varepsilon\to 0}\mathbb{E}||\beta^\varepsilon-\beta||_\alpha^r=0.$$

4.2 Example two. Moderate Deviations Principle for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficient

In this section we establish the MDP for the stochastic Cahn-Hilliard equation (4.1). Consider the process $\Theta^\varepsilon(t,x)$ such that

$$\Theta^\varepsilon(t,x) := \left(\frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}a(\varepsilon)}\right)(t,x). \tag{4.10}$$

In this section, we study the LDP for $\Theta^\varepsilon(t,x)$ defined by (4.10) as $\varepsilon \to 0$ with $1 < a(\varepsilon) < \frac{1}{\sqrt{\varepsilon}}$.

Theorem 6: *The process* $\{\Theta^\varepsilon(t,x)\}_{\varepsilon>0}$ *defined by* (4.10) *obeys a LDP on the space* $\mathcal{C}^\alpha([0,1]\times\mathcal{O})$*, with speed* $a^2(\varepsilon)$ *and rate function* $\mathcal{J}_{M.D.P}(.)$ *such that :*

$$\mathcal{J}_{M.D.P}(g) = \inf_{g=\mathcal{G}^0(u_0,\mathcal{I}(h))}\left\{\frac{1}{2}\int_0^T\int_0^\pi\int_0^\pi\int_0^\pi \dot{h}^2(t,x)dtdx_1dx_2dx_3\right\}$$

and $+\infty$ *otherwise.*

Proof of Theorem 6: It is sufficient to prove that

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\left(\frac{|\beta^\varepsilon - \beta^0|_\alpha}{a(\varepsilon)} > \delta\right) = -\infty \quad , \quad \forall \delta > 0.$$

Recall the decomposition in the proof of Theorem 5

$$\beta^\varepsilon(t,x) - \beta^0(t,x) = m_1^\varepsilon(t,x) + m_2^\varepsilon(t,x) + m_2^\varepsilon(t,x).$$

For any q in $(\frac{3}{2},3)$, $\frac{1}{p}+\frac{1}{q}=1$, and $x,y\in\mathcal{O}$, $0\le s\le t\le T$, by Hölder's inequality, 4 in Lemma 1 and (2.3), we have

$$\left|m_2^\varepsilon(t,x) - m_2^\varepsilon(t,y)\right|^p \quad \le \quad 16|x-y|^{\frac{3-q}{q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}. \tag{4.11}$$

Similarly, in view of 5 and 6, it follows that for $0\le s\le t\le T$,

$$\left|m_2^\varepsilon(t,y) - m_2^\varepsilon(s,y)\right|^p \quad \le \quad 32|t-s|^{\frac{3-q}{2q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}. \tag{4.12}$$

Putting together (4.11), (4.12), we have

$$\left|m_2^\varepsilon(t,y) - m_2^\varepsilon(s,y)\right|^p \le C(K_f)(|t-s|+|x-y|^2)^{\frac{3-q}{2q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}.$$

Choosing $q\in(\frac{3}{2},3)$, such that $\alpha = 3-q/2q$ and noticing that $||\beta^\varepsilon - \beta^0||_\infty^u \le (1+u)^\alpha|\beta^\varepsilon-\beta^0|_\alpha^u$, we obtain that

$$|m_2^\varepsilon|_\alpha^t \le C(K_f)\left(\int_0^t ((1+u)^\alpha|\beta^\varepsilon - \beta^0|_\alpha^u)^p du\right)^{\frac{1}{p}}.$$

Thus, for $t\in[0,1]$, we have

$$(|\beta_t^\varepsilon - \beta_t^0|_\alpha^t)^p \le C(p,T,K_f)\left[\left(|m_1^\varepsilon(t)|_\alpha^t + |m_3^\varepsilon(t)|_\alpha^t\right)^p + \int_0^t (|\beta^\varepsilon - \beta^0|_\alpha^s)^p ds\right].$$

Applying Gronwall's Lemma to $\Psi(t) = (|\beta_t^\varepsilon - \beta_t^0|_\alpha^t)^p$, we have

$$(|\beta_t^\varepsilon - \beta_t^0|_\alpha)^p \le C(p,T,K_f)\left[\left(|m_1^\varepsilon(t)|_\alpha^t + |m_3^\varepsilon(t)|_\alpha^t\right)^p\right]e^{C(p,T,K_f)T}. \tag{4.13}$$

By (4.12) and (4.13), it is sufficient to prove that for any $\delta>0$,

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\left(\frac{|m_i^\varepsilon(t)|_\alpha^T}{a(\varepsilon)} > \delta\right) = -\infty \qquad\qquad i=1,3.$$

Step 1. For any $\varepsilon > 0, \eta > 0$ we have

$$\begin{aligned} \mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta\big) &\leq \mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta, |u^\varepsilon - u^0|_\infty^T < \eta\big) \\ &+ \mathbb{P}(|u^\varepsilon - u^0|_\infty^T \geq \eta) \end{aligned} \tag{4.14}$$

By 4 and 6 in Lemma 1, $\big(\sum_{i=0}^\infty e^{-\mu_i^2(t-s)} w_i(x) w_i(y)\big).1_{[u\leq t]}$ satisfies (3.12)(see Lemma 3) for $\alpha_0 = \frac{1}{2}$. Applying Lemma 3, we have

$$F(t,x,u,z) = \big(\sum_{i=0}^\infty e^{-\mu_i^2(t-s)} w_i(x) w_i(z)\big) 1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, M = a(\varepsilon)\delta,$$

$\rho = \eta K_\sigma, Y^*(t,x) = \big(u^0(t,x) - u^\varepsilon(t,x)\big) 1_{||u^\varepsilon - u^0||_\infty^T > \eta}$
we obtain that for all ε sufficiently small such that $a(\varepsilon)\delta \geq \rho C C(\alpha, \frac{1}{2})$

$$\mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta, ||u^\varepsilon - u^0||_\infty^T < \eta\big) \leq (\sqrt{2}T^2 + 1) \exp\big(-\frac{a^2(\varepsilon)\delta^2}{\eta^2 K_\sigma^2 C C^2(\alpha, \frac{1}{2})}\big). \tag{4.15}$$

Since u^ε satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times\mathcal{O})$

$$\begin{aligned} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(||u^\varepsilon - u^0||_\infty^T \geq \eta) &\leq \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(||u^\varepsilon - u^0||_\alpha \geq \eta) \\ &\leq -\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}. \end{aligned}$$

In this case, the good rate function $\mathcal{I} = \{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}$ has compact level sets, the "$\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}$" is obtained at some function f_0. Because $\mathcal{I}(f) = 0$ if and only if $f = u^0$, we conclude that

$$-\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\} < 0.$$

For $a(\varepsilon) \to \infty$, $\sqrt{\varepsilon} a(\varepsilon) \to 0$, we have

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon) \log \mathbb{P}\big(||u^\varepsilon - u^0||_\infty^T \geq \eta\big) = -\infty. \tag{4.16}$$

Since $\eta > 0$ is arbitrary, putting together (4.14), (4.15) and (4.16), we obtain

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon) \log \mathbb{P}\left(\frac{||m_3^\varepsilon||_\alpha}{a(\varepsilon)} \geq \delta\right) = -\infty. \tag{4.17}$$

Step 2. For the first term $m_1^\varepsilon(t)$, let

$$m_1^\varepsilon(t,x) = \int_0^t \int_{\mathcal{O}} \Delta G_{t-s}(x,y) \mathfrak{M}^\varepsilon(s,y) ds dy,$$

where

$$\begin{aligned} \mathfrak{M}^\varepsilon(s,y) \quad = 4 \quad & \Bigg(\Bigg(\frac{(u^\varepsilon(s,y))^3 - (u^0(s,y))^3}{\sqrt{\varepsilon}}\Bigg) - \Bigg(\frac{u^\varepsilon(s,y) - u^0(s,y)}{\sqrt{\varepsilon}}\Bigg) \\ & -\big(3(u^0(s,y))^2 - 1\big)\beta^\varepsilon(s,y)\Bigg) \end{aligned}$$

as stated in the proof of Theorem 5, we have

$$||\mathfrak{M}^\varepsilon||_\infty^T \leq C \frac{(||u^\varepsilon - u^0||_\infty^T)^2}{\sqrt{\varepsilon}}.$$

However, by the Hölder's continuity of Green function G, it is easy to prove that, for any $\alpha \in (0, \frac{1}{4})$

$$|m_2^\varepsilon|_\alpha^T \leq C(\alpha, T) ||\mathfrak{M}^\varepsilon||_\infty^T.$$

From the proof of proposition 1, we obtain that

$$||u^\varepsilon - u^0||_\infty^T \leq C(T) ||\widetilde{m}_2^\varepsilon||_\infty^T.$$

where

$$\widetilde{m}_2^\varepsilon(t,x) = \sqrt{\varepsilon \int_0^t \int_{\mathcal{O}} \Delta G_{t-s}(x,y) u^\varepsilon(s,y) W(dsdy)}.$$

Applying lemma 3, we have

$$F(t,x,u,z) = G_{t-u}(x,z)1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, \rho = \sqrt{\varepsilon} K(1 + ||u^T||_\infty^T + \eta)$$
$$Z^*(t,x) = \sqrt{\varepsilon}(1 - u^\varepsilon(t,x))1_{[||u^\varepsilon||_\infty^T < ||u^0||_\infty^T + \eta]},$$

for any $\eta > 0$, we obtain that for all ε is sufficiently small such that $M \geq \sqrt{\varepsilon}(1 + ||u^T||_\infty^T + \eta) C C(\alpha, \frac{1}{2})$,

$$\mathbb{P}(||\widetilde{m}_2^\varepsilon||_\infty^T \geq M, ||u^\varepsilon||_\infty^T < ||u^0||_\infty^T + \eta)$$
$$\leq (\sqrt{2}T^2 + 1) \exp\bigg(- \frac{M^2}{\varepsilon K^2 C C^2(\alpha, \frac{1}{2})(1 + ||u^0||_\infty^T + \eta)^2} \bigg).$$

For the same raison as (4.11), we obtain

$$\limsup_{\varepsilon \to 0} a^{-2}(\varepsilon) \log \mathbb{P}(||u^\varepsilon||_\infty^T \geq ||u^0||_\infty^T + \eta)$$
$$\leq \limsup_{\varepsilon \to 0} a^{-2}(\varepsilon) \log \mathbb{P}(||u^\varepsilon - u^0||_\infty^T \geq \eta) = -\infty.$$

For any $\eta > 0$, by Bernstein's inequality and the continuity of σ, we have

$$\limsup_{\varepsilon \to 0} a^{-2}(\varepsilon) \log \mathbb{P}\bigg(\frac{|m_1^\varepsilon(t)|_\alpha^T}{a(\varepsilon)} \geq \delta \bigg)$$

$$\begin{aligned}
\leq \quad & \limsup_{\varepsilon \to 0} a^{-2}(\varepsilon) \log \mathbb{P}\bigg(\bigg(||\widetilde{m}_2^\varepsilon||_\infty^T \bigg)^2 \geq \frac{\sqrt{\varepsilon} a(\varepsilon)\delta}{C(\alpha, T, K_f, C)} \bigg) \\
\leq \quad & \limsup_{\varepsilon \to 0} a^{-2}(\varepsilon) \log \bigg[\mathbb{P}\bigg((||\widetilde{m}_2^\varepsilon(t)||_\infty^T)^2 \geq \frac{\sqrt{\varepsilon} a(\varepsilon)\delta}{C(\alpha, T, K_f, C)}, \\
& ||u^\varepsilon|| < ||u^0||_\infty^T + \eta \bigg) + \mathbb{P}(||u^\varepsilon|| \geq ||u^0||_\infty^T + \eta) \bigg] \\
\leq \quad & \bigg(\limsup_{\varepsilon \to 0} \frac{-\delta}{\sqrt{\varepsilon} a(\varepsilon) C(\alpha, T, K_f, C) K^2 C C^2(\alpha, \frac{1}{2})(1 + ||u^0||_\infty^T + \eta)^2} \bigg) \\
& \vee \bigg(\limsup_{\varepsilon \to 0} h^{-2}(\varepsilon) \log \mathbb{P}(||X_{X_0}^\varepsilon|| \geq ||X_{X_0}^0||_\infty^T + \eta) \bigg) = -\infty.
\end{aligned}$$

5 Conclusion

In this paper, we have proved a CLT and a MDP for a perturbed stochastic Cahn-Hilliard equation in Hölder space by using the exponential estimates in the space of Hölder continuous functions and the Garsia-Rodemich-Rumsey's lemma. We can also examine the same situation in Besov-Orlicz space.

Acknowledgement

The authors wish to thank the referees and editors for their valuable comments and suggestions which led to improvements in the document.

Declaration

The authors declare no conflict of interest.

REFERENCES

[1] Ben Arous, G., & Ledoux, M. (1994). Grandes déviations de Freidlin-Wentzell en norme hölderienne. *Séminaire de probabilités de Strasbourg*, 28, 293-299.

[2] Boulanba, L., & Mellouk, M. (2020). Large deviations for a stochastic Cahn–Hilliard equation in Hölder norm. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 23(02), 2050010.

[3] Cahn, J. W., & Hilliard, J. E. (1971). Spinodal decomposition: A reprise. *Acta Metallurgica*, 19(2), 151-161.

[4] Cahn, J. W., & Hilliard, J. E. (1958). Free energy of a nonuniform system. I. Interfacial free energy. *The Journal of chemical physics*, 28(2), 258-267.

[5] Cardon-Weber, C. (2001). Cahn-Hilliard stochastic equation: existence of the solution and of its density. *Bernoulli*, 777-816.

[6] Chenal, F., & Millet, A. (1997). Uniform large deviations for parabolic SPDEs and applications. *Stochastic Processes and their Applications*, 72(2), 161-186.

[7] Freidlin, M. I. (1970). On small random perturbations of dynamical systems. *Russian Mathematical Surveys*, 25(1), 1-55.

[8] Li, R., & Wang, X. (2018). Central limit theorem and moderate deviations for a stochastic Cahn-Hilliard equation. *arXiv preprint arXiv:1810.05326*.

[9] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. *Lecture notes in mathematics*, 265-439.

[10] Wang, R., & Zhang, T. (2015). Moderate deviations for stochastic reaction-diffusion equations with multiplicative noise. *Potential Analysis*, 42, 99-113.