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Stabilization of the transmission Schrödinger equation with boundary time-varying delay.

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ABSTRACT: We consider a system of transmission of the Schrödinger equation with Neumann feedback control that contains a time-varying delay term and that acts on the exterior boundary. Using a suitable energy function and a suitable Lyapunov functionnal, we prove under appropriate assummptions that the solutions decay exponentially.

Keywords: Schrödinger equation, transmission problem, time-varying delay, exponential stability, boundary stabilization.

MSC: 35Q93, 93D15

1 Introduction and statement of the exponential stability result

The analysis of the effect of time delays in feedback stabilization of control systems described by partial differential equations has received considerable attention in the literature, (see [2] and the references therein). It is by now well known that certain hyperbolic systems which are stabilized by feedback controls become unstable when arbitrary small time delays occur in these controls [9], [8]. Xu et al [22] established sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the linear boundary feedback. Nicaise and Pignotti [18] extended this result to the multi-dimensional wave equation with a delay term in the linear boundary or internal feedback; they further underline some instability phenomenon. Rebiai and Sidiali [20] considered a multi-dimensional transmission wave equation with a Neumann feedback control that contains a discrete delay term and that acts on the exterior boundary. They showed, under some assumptions, that some energy function of the solution decays exponentially. To obtain this result, they used multipliers technique and compactness uniqueness argument..

Stabilization problems for the Schrödinger equation with time delay have also been studied and many nice results have been obtained. Guo and Yang [11] developed an observer-predictor scheme to stabilize the 1-d Schrödinger equation with time delay in the observation. Guo and Mei [10] generalized this scheme to a multi-dimensional Schrödinger equation with partial Dirichlet control and collocated observation with time delay. Yang and Yao [23] used a similar approach to stabilize a 1-d Schrödinger equation with variable coefficients and boundary output time delay. Cui et al [6] designed a dynamical feedback control based on a partial state predictor to stabilize the 1-d Schrödinger equation with a time delay in the boundary input. Cui et al [7] adopted a ”detecting-predicting” procedure to stabilize the 1-d Schrödinger equation with a distributed time delay in the boundary input. Nicaise and Rebiai [17] considered the multi-dimensional Schrödinger equation with a discrete time delay term in the boundary or internal feedbacks. In both cases,

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they showed that if the coefficient of the delayed feedback term is smaller than the one of the undelayed damping term, then the solution decays exponentially in an appropriate functional space. These results are obtained by proving some observability estimates. In the opposite case, they constructed a sequence of delays that destabilize these systems. Chen et al [4] used the concept of system equivalence to design a feedback control for the multi-dimensional Schrödinger equation with internal delayed control.

Motivated by [17] and [20], we present in this paper a stability result for the transmission Schrödinger equation with time-varying delay term in the boundary feedback. Stabilization problems for the undelayed transmission Schrödinger equation have been investigated in [5] and [1]. In [5], the authors proved exponential decay of the energy of the solutions under linear boundary dissipation in the Neumann boundary condition by adopting a frequency domain approach which is based upon a resolvent criterion. Reference [1] gives a uniform stabilization result with a dissipative feedback acting in the Dirichlet boundary condition by establishing exact controllability of the corresponding open-loop system.

Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let Γ_0 with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \overline{\Gamma}_0 \cap \overline{\Gamma}_2 = \emptyset$ be a regular hypersurface of class C^2 which separates Ω into two domains Ω_1 and Ω_2 such that $\Gamma_1 \subset \partial\Omega_1$ and $\Gamma_2 \subset \partial\Omega_2$. , and assume that there exists $x_0 \in \mathbb{R}^n$ such that for $m(x) = x - x_0$, we have:

$$m(x).\nu(x) \leq 0 \qquad \text{on } \Gamma_1 \text{ and on } \Gamma_0, \tag{1.1}$$

$$m(x).\nu(x) \geq \delta > 0 \qquad \text{on } \Gamma_2, \tag{1.2}$$

where ν is the unit normal on Γ or Γ_0 pointing towards Ω or Ω_1.

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the Schrödinger equation with a time-varying delay term in the boundary conditions:

$$\partial_t y_k(x,t) - i a_k \Delta y_k(x,t) = 0, \qquad \text{in } \Omega_k \times (0,+\infty), k = 1,2, \tag{1.3}$$

$$y_k(x,0) = y_{0k}(x) \qquad \text{in } \Omega_k, k = 1,2, \tag{1.4}$$

$$y_1(x,t) = 0, \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{1.5}$$

$$\frac{\partial y_2(x,t)}{\partial \nu} = -\alpha \partial_t y_2(x,t) - \beta \partial_t y_2(x, t-\tau(t)), \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.6}$$

$$y_1(x,t) = y_2(x,t), \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{1.7}$$

$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu}, \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{1.8}$$

$$\partial_t y_2(x, t-\tau(0))) = f_0(x, t-\tau(0)), \qquad \text{on } \Gamma_2 \times (0,\tau(0)), \tag{1.9}$$

where:

- α and β are positive constants,
- y_{01}, y_{02}, f_0 are the initial data which belong to suitable spaces,
- $\tau(.)$ is the time-varying which is as in [19] subject to the following assumptions:
  There exist positive constants $\widehat{\tau}$ and $\widetilde{\tau}$ such that

$$0 < \widehat{\tau} \leq \tau(t) \leq \widetilde{\tau} \ \text{ for all } t > 0, \tag{1.10}$$

$$\tau'(t) \leq d < 1 \qquad \text{for all } t > 0, \tag{1.11}$$

$$\tau(.) \in W^{2,\infty}([0,T]). \tag{1.12}$$

In this paper, we introduce a suitable energy function and a suitable Lyapunov functionnal to prove that solutions of (1.3)-(1.9) decay exponentially in an appropriate Hilbert space. A similar approach has been adopted in [19] to study the stability of the multi-dimensional wave equation with a time-varying delay term in the boundary feedback.

To state our stability result, we assume as in [19] that

$$\alpha\sqrt{1-d} > \beta, \tag{1.13}$$

and define the energy of a solution

$$y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,+\infty), \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,+\infty), \end{cases}$$

of $(1.3)-(1.9)$ by

$$E(t)=\frac{a_1}{2}\int_{\Omega_1}|\nabla y_1(x,t)|^2\,dx+\frac{a_2}{2}\int_{\Omega_2}|\nabla y_2(x,t)|^2\,dx+\frac{\xi}{2}\tau(t)\int_{\Gamma_2}\int_0^1|\partial_t y_2(x,t-\rho\tau(t))|^2\,d\rho\,d\Gamma, \quad (1.14)$$

where

$$\frac{a_2\beta}{\sqrt{1-d}}<\xi<2\alpha a_2-\frac{a_2\beta}{\sqrt{1-d}}. \quad (1.15)$$

The main result of this paper can be stated as follows.

Theorem 1.1. *In addition to* (1.1), (1.2), *(1.10), (1.11), (1.12) and* (1.13), *assume that*

$$a_1>a_2.$$

Then there exist constants $M\geq 1$ *and* $\omega>0$ *such that*

$$E(t)\leq Me^{-\omega t}E(0),$$

for any regular solution of $(1.3)-(1.9)$.

Theorem 1.1 is proved in Section 3. In Section 2, we study existence, uniqueness and regularity of solutions for system $(1.3)-(1.9)$ using semigroup theory.

2 Well-posedness result

Inspired by [18], we introduce the auxiliary variable

$$z(x,\rho,t)=\partial_t y_2(x,t-\tau(t)\rho), x\in\Gamma_2, \rho\in(0,1), t>0. \quad (2.1)$$

With this new unknown, problem (1.3)-(1.9) is equivalent to

$$\begin{aligned}
&\partial_t y_k(x,t)-ia_k\Delta y_k(x,t)=0, && \text{in } \Omega_k\times(0,+\infty), k=1,2, && (2.2)\\
&y_k(x,0)=y_{0k}(x), && \text{in } \Omega_k, k=1,2, && (2.3)\\
&y_1(x,t)=0, && \text{on } \Gamma_1\times(0,+\infty), && (2.4)\\
&\tau(t)\partial_t z(x,\rho,t)+(1-\tau'(t)\rho)\partial_\rho z(x,\rho,t)=0, && \text{in } \Gamma_2\times(0,1)\times(0,+\infty) && (2.5)\\
&\frac{\partial y_2(x,t)}{\partial\nu}=-i\alpha a_2\Delta y_2(x,t)-\beta z(x,1,t), && \text{on } \Gamma_2\times(0,+\infty), && (2.6)\\
&y_1(x,t)=y_2(x,t), && \text{on } \Gamma_0\times(0,+\infty), && (2.7)\\
&a_1\frac{\partial y_1(x,t)}{\partial\nu}=a_2\frac{\partial y_2(x,t)}{\partial\nu}, && \text{on } \Gamma_0\times(0,+\infty), && (2.8)\\
&z(x,0,t)=\partial_t y_2(x,t), && \text{on}\Gamma_2\times(0,+\infty), && (2.9)\\
&z(x,\rho,0)=f_0(x,-\rho\tau(0)), && \text{in } \Gamma_2\times(0,1). && (2.10)
\end{aligned}$$

Let

$$\mathcal{V}=\{(u_1,u_2)\in H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2); u_1=u_2 \text{ on } \Gamma_0\}.$$

The space for well-posedness of (2.2)-(2.10) is taken to be the space

$$\mathcal{H}=\mathcal{V}\times L^2(\Gamma_2;L^2(0,1)).$$

$\mathcal{H}$ is a Hilbert space with the following inner product

$$\left\langle\begin{pmatrix}u_1\\u_2\\z\end{pmatrix};\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{z}\end{pmatrix}\right\rangle=a_1\int_{\Omega_1}\nabla u_1(x).\nabla\widetilde{u}_1(x)dx+a_2\int_{\Omega_2}\nabla u_2(x).\nabla\widetilde{u}_2(x)dx+\xi\int_{\Gamma_2}\int_0^1 z(x,\rho)\widetilde{z}(x,\rho)d\rho d\Gamma.$$

In $\mathcal{H}$, define a linear operator $A(t)$ by

$$A(t)(u_1,u_2,z)^T=(ia_1\Delta u_1, ia_2\Delta u_2, \frac{\tau'(t)\rho-1}{\tau(t)}\partial_\rho z)^T, \quad (2.11)$$

$$\begin{aligned} D(A(t)) \;=\; & \{(u_1,u_2,z)^T \in \mathcal{V}\times L^2(\Gamma_2;H^1(0,1)); \Delta u_1 \in H^1_{\Gamma_1}(\Omega_1), \Delta u_2 \in H^1(\Omega_2), \\ & a_1\frac{\partial u_1}{\partial \nu} = a_2\frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0, a_1\Delta u_1 = a_2\Delta u_2 \text{ on } \Gamma_0, z(.,0) = ia_2\Delta u_2 \text{ on } \Gamma_2, \\ & \frac{\partial u_2}{\partial \nu} = -\alpha z(.,0)) - \beta z(.,1), \text{ on } \Gamma_2\}. \end{aligned} \tag{2.12}$$

Notice that for $(u_1,u_2,z)\in D(A(t))$, we have the following boundary regularity:

- $\Delta u_k|_{\partial\Omega_k} \in H^{1/2}(\partial\Omega_k), k = 1, 2,$ (trace theorem),
- $\frac{\partial u_1}{\partial \nu}|_{\partial\Omega_1} \in H^{-1/2}(\partial\Omega_1)$, $\frac{\partial u_2}{\partial \nu}|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$ (see e.g., [14] p. 71, Theorem. 3.8.1]),
- $\frac{\partial u_2}{\partial \nu}|_{\Gamma_2} \in L^2(\Gamma_2)$ since $z \in L^2(\Gamma_2)$.

Using the operator $A(t)$, we rewrite $(2.2)-(2.10)$ as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = A(t)Y(t), \\ \quad Y(0) = Y_0, \end{cases} \tag{2.13}$$

where

$$Y(t) = (y,z)^T \text{ and } Y_0 = (y0(x), f_0(.,-.\tau(0)))^T.$$

Notice that problem (2.13) is equivalent to

$$\begin{cases} \frac{d}{dt}\widetilde{Y}(t) = \widetilde{A}(t)\widetilde{Y}(t), \\ \quad \widetilde{Y}(0) = Y_0, \end{cases} \tag{2.14}$$

where

$$\widetilde{A}(t) = A(t) - \kappa(t)I, \kappa(t) = \frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}, \tag{2.15}$$

in the sense that if $\widetilde{Y}(t)$ is a solution of (2.14) then $Y(t) = e^{\theta(t)}\widetilde{Y}(t)$ where $\theta(t) = \int_0^t \kappa(s)ds$ is a solution of (2.13).

To establish existence and uniqueness of solutions for problem (2.14), we employ the result stated next [12], [13].

Theorem 2.1. *Let $A(t) : D(A(t)) \subset \mathcal{H} \to \mathcal{H}$ be a time-varying linear operator such that :*

i. *$D(A(t))$ is independent of t,*
ii. *$D(A(0))$ is a dense subset of $\mathcal{H}$,*
iii. *For all $t\in[0,T]$ $A(t)$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}$,*
iv. *The family $\mathcal{A} = \{A(t) : t \in [0,T]\}$ is stable with stability constants C and m independents of t, i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $A(t)$ satisfies the estimate*

$$\|S_t(s)f\|_{\mathcal{H}} \leq Ce^{ms}\|f\|_{\mathcal{H}},$$

for all $f\in\mathcal{H}$ and $s\geq 0$,

v. *$\frac{d}{dt}A(t) \in L^\infty_*([0,T], B(D(A(0)),\mathcal{H})$, which is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(D(A(0))),\mathcal{H})$ of bounded operators from Y into $\mathcal{H}$.*

Then problem

$$\begin{cases} \frac{d}{dt}U(t) = A(t)U(t), \\ \quad U(0) = U_0, \end{cases}$$

has a unique solution

$$U \in C([0,T], D(A(t))) \cap C^1([0,T],\mathcal{H}),$$

for any initial datum in $D(A(0))$.

Below, we prove that the conditions required by Theorem 2.1 are met by the operator $\widetilde{A}(t)$. Since $D(\widetilde{A}(t)) = D(A(t))$, then it follows from (2.12) that

$$D(\widetilde{A}(t)) = D(\widetilde{A}(0)), \tag{2.16}$$

that is the domain of $\widetilde{A}(t)$ is independent of t.

Proposition 2.2. *For each fixed $t \in [0, T]$, the operator $\widetilde{A}(t)$ generates a C_0-semigroup $\widetilde{S}_t(s)$ on $\mathcal{H}$.*

Proposition 2.2 follows as a consequence of Lemmas 2.3 and 2.4 which are stated below.

Lemma 2.3. *$D(\widetilde{A}(0))$ is dense in $\mathcal{H}$.*

Proof. It is sufficient to show that $D(A(0))$ is a dense subset of $\mathcal{H}$. We proceed as in [19]. Let $(f_1, f_2, g)^T \in \mathcal{H}$ be orthogonal to all elements of $D(A(0)), i.e.$

$$a_1 \int_{\Omega_1} \nabla y_1(x).\nabla \overline{f_1}(x)dx + a_2 \int_{\Omega_2} \nabla y_2(x).\nabla \overline{f_2}(x)dx + \int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma = 0, \tag{2.17}$$

for all $(y_1, y_2, z)^T \in D(A(0))$.

For $y_1 = 0, y_2 = 0$ and $z \in \mathcal{D}(\Gamma_2 \times (0,1)); (y_1, y_2, z)^T \in D(A(0))$, and

$$\int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma = 0.$$

Since $\mathcal{D}(\Gamma_2 \times (0,1))$ is dense in $L^2(\Gamma_2, L^2(0,1))$, we conclude that $g = 0$.
In the same manner, we obtain $y_2 = 0$ if we take in (2.17), $f_1 = 0$, $z = 0$ and $f_2 \in \mathcal{D}(\Omega_2)$. Therefore, the identity (2.17) is reduced to

$$\int_{\Omega_1} \nabla \overline{y_1}(x).\nabla \overline{f_1}(x)dx = 0 \ \forall (y_1, y_2, z) \in D(A(0)). \tag{2.18}$$

By taking in (2.18), $y_2 = 0$ and $z = 0$, we get

$$\int_{\Omega_1} \nabla y_1(x).\nabla \overline{f_1}(x)dx = 0 \qquad \text{for all } (y_1, 0, 0)^T \in D(A(0)). \tag{2.19}$$

But $(y_1, 0, 0)^T \in D(A(0))$ if and only if $y_1 \in \mathcal{D} = \{f \in \mathcal{V}; \Delta u_1 \in \mathcal{V}, \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_0\}$. Since $\mathcal{D}_1 = \{f \subset \mathcal{V} \cap H^2(\Omega_1) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_0\} \subset \mathcal{D}$ and $\mathcal{D}_1$ is dense in $\mathcal{V}$. Then $\mathcal{D}$ is dense in $\mathcal{V}$. Combining this fact with (2.19), we conclude that $f_1 = 0$. □

Lemma 2.4. *Define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product*

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} \right\rangle_t = a_1 \int_{\Omega_1} \nabla y_1(x).\nabla \overline{f_1}(x)dx + a_2 \int_{\Omega_2} \nabla y_2(x).\nabla \overline{f_2}(x)dx +$$
$$\xi\tau(t)\int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma.$$

Then $\widetilde{A}(t)$ is dissipative for fixed t.

Proof. Because of (2.15), it is sufficient to show that $A(t)$ is dissipative. Let $Y = (y_1, y_2, z)^T \in D(A(t))$. Then

$$\begin{aligned}\Re \langle Y, A(t)Y\rangle_t &= -\Re\{a_1^2 i \int_{\Omega_1} \nabla y_1(x).\nabla(\Delta \overline{y_1(x)})dx\} - \Re\{a_2^2 i \int_{\Omega_2} \nabla y_2(x).\nabla(\Delta \overline{y_2(x)})dx\} + \\ &\quad \Re\xi\tau(t)\int_{\Gamma_2}\int_0^1 z(x,\rho)\frac{\tau'(t)\rho - 1}{\tau(t)}\overline{\partial_\rho z(x,\rho)}d\rho d\Gamma.\end{aligned} \tag{2.20}$$

Applying Green's theorem to the first two integrals on the right-hand side of (2.20) and using the fact that the normal vector on Γ_0 is oriented towards the interior of Ω_1, we obtain

$$\begin{aligned}
&-\Re\{a_1^2 i\int_{\Omega_1}\nabla y_1(x).\nabla(\Delta\overline{y_1(x)})dx\} - \Re\{a_2^2 i\int_{\Omega_2}\nabla u_2(x).\nabla(\Delta\overline{y_2(x)})dx\} = -\\
&\Re\{a_1^2 i\int_{\Gamma_1}\frac{\partial y_1(x)}{\partial\nu}\Delta\overline{y_1(x)}d\Gamma + a_1^2 i\int_{\Gamma_0}\frac{\partial y_1(x)}{\partial\nu}\Delta\overline{y_1(x)}d\Gamma - a_1^2 i\int_{\Omega_1}|\Delta y_1(x)|^2\,dx\ \}-\\
&\Re\{a_2^2 i\int_{\Gamma_2}\frac{\partial y_2(x)}{\partial\nu}\Delta\overline{y_2(x)}d\Gamma - a_2^2 i\int_{\Gamma_0}\frac{\partial y_2(x)}{\partial\nu}\Delta\overline{y_2(x)}d\Gamma - a_2^2 i\int_{\Omega_2}|\Delta y_2(x)|^2\,dx\}.
\end{aligned}\tag{2.21}$$

Note that the integrals over Γ_1 (resp.Γ_0) on the right-hand side of (2.21) are to be interpreted in the sense of duality pairing between $H^{1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_1)$ (resp. $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$)
(2.21) together with (2.12) yields

$$\begin{aligned}
&-\Re\{a_1^2 i\int_{\Omega_1}\nabla y_1(x).\nabla(\Delta\overline{y_1(x)})dx\} - \Re\{a_2^2 i\int_{\Omega_2}\nabla y_2(x).\nabla(\Delta\overline{y_2(x)})dx\} = -\\
&a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2}z(x,1)\overline{z}(x,0)d\Gamma.
\end{aligned}\tag{2.22}$$

Integrating by parts in ρ the third integral on the right-hand side of (2.20), we get

$$\begin{aligned}
&\Re\xi\int_{\Gamma_2}\int_0^1 z(x,\rho)(\tau'(t)\rho-1)\overline{\partial_\rho z(x,\rho)}d\rho d\Gamma = \frac{\xi}{2}\int_{\Gamma_2}\{|z(x,1)|^2(\tau'(t)-1)+|z(x,0)|^2\}d\Gamma-\\
&\frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2\,d\rho d\Gamma.
\end{aligned}\tag{2.23}$$

Inserting (2.22) and (2.23) into (2.20) results in

$$\begin{aligned}
&\Re\langle Y, A(t)Y\rangle_t = -a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2}z(x,1)\overline{z}(x,0)d\Gamma-\\
&\frac{\xi}{2}\int_{\Gamma_2}\{|z(x,1)|^2(1-\tau'(t))-|z(x,0)|^2\}d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2\,d\rho d\Gamma.
\end{aligned}$$

from which follows after using the Cauchy-Schwarz inequality and the assumption (1.11),

$$\begin{aligned}
&\Re\langle Y, A(t)Y\rangle_t \le -(a_2\alpha-\frac{\xi}{2}-\frac{a_2\beta}{2\sqrt{1-d}})\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma-\\
&(\frac{\xi(1-d)}{2}-\frac{a_2\beta\sqrt{1-d}}{2})\int_{\Gamma_2}|z(x,1)|^2\,d\Gamma-\kappa(t)\,\|Y\|_t^2\,.
\end{aligned}\tag{2.24}$$

□

Lemma 2.5. *The operator $\widetilde{A}(t)$ is maximal for each fixed t.*

Proof. Since $\kappa(t)>0$, then the maximality of $\widetilde{A}(t)$ follows from that of $A(t)$. To this end, let $(f_1,f_2,g)^T\in\mathcal{H}$, and consider for some $\lambda>0$ the equation

$$(\lambda I - A(t))Y = (f_1,f_2,g)^T,$$

where $Y = (y_1, y_2, z)^T \in D(A(t))$ or equivalently

$$\lambda y_k(x) - ia_k \Delta y_k(x) = f_k(x), \quad \text{in } \Omega_k, k = 1, 2, \tag{2.25}$$
$$\lambda z(x,\rho) + \frac{1-\tau'(t)\rho}{\tau(t)} \partial_\rho z(x,\rho) = g(x,\rho), \quad \text{on } \Gamma_2 \times (0,1), \tag{2.26}$$
$$y_1(x) = 0, \quad \text{on } \Gamma_1, \tag{2.27}$$
$$\frac{\partial y_2(x)}{\partial \nu} = -\alpha z(x,0) - \beta z(x,1), \quad \text{on } \Gamma_2, \tag{2.28}$$
$$y_1(x) = y_2(x), \quad \text{on } \Gamma_0, \tag{2.29}$$
$$a_1 \frac{\partial y_1(x)}{\partial \nu} = a_2 \frac{\partial y_2(x)}{\partial \nu}, \quad \text{on } \Gamma_0, \tag{2.30}$$
$$a_1 \Delta y_1(x) = a_2 \Delta y_2(x), \quad \text{on } \Gamma_0. \tag{2.31}$$

We can determine z once we have found (y_1, y_2) with the appropriate regularity. Indeed, from (2.26) and (2.12), we have

$$\begin{cases} \partial_\rho z(x,\rho) = \frac{\lambda\tau(t)}{1-\tau'(t)\rho} z(x,\rho) + \frac{\tau(t)}{1-\tau'(t)\rho} g(x,\rho), & x \in \Gamma_2, \rho \in (0,1), \\ z(x,0) = ia_2 \Delta y_2(x), & x \in \Gamma_2. \end{cases}$$

The unique solution of the above initial value problem is given by

$$z(x,\rho) = e^{-\lambda\rho\tau(t)} z(x,0) + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^\rho e^{\lambda s \tau(t)} g(x,s) ds,$$

if $\tau'(t) = 0$ and by

$$z(x,\rho) = z(x,0) \exp\left(\frac{\lambda\tau(t)\ln(1-\tau'(t)\rho)}{\tau'(t)}\right) + \exp\left(\frac{\lambda\tau(t)\ln(1-\tau'(t)\rho)}{\tau'(t)}\right) \int_0^\rho \frac{g(x,s)\tau(t)}{1-\tau'(t)s} \exp\left(\frac{-\lambda\tau(t)\ln(1-\tau'(t)s)}{\tau'(t)}\right) ds,$$

if $\tau'(t) \neq 0$.
In particular

$$z(x,1) = e^{-\lambda\tau(t)} z(x,0) + v(x), \quad x \in \Gamma_2, \tag{2.32}$$
$$z(x,1) = z(x,0) \exp\left(\frac{\lambda\tau(t)\ln(1-\tau'(t))}{\tau'(t)}\right) + v(x), \quad x \in \Gamma_2,$$

where $v(.), w(.) \in L^2(\Gamma_2)$ and are defined by

$$v(x) = \tau(t) e^{-\lambda 1 \tau(t)} \int_0^1 e^{\lambda s \tau(t)} g(x,s) ds,$$
$$w(x) = \exp\left(\frac{\lambda\tau(t)\ln(1-\tau'(t))}{\tau'(t)}\right) \int_0^1 \frac{g(x,s)\tau(t)}{1-\tau'(t)s} \exp\left(\frac{-\lambda\tau(t)\ln(1-\tau'(t)s)}{\tau'(t)}\right) ds.$$

From (2.25), we have

$$\lambda y_1(x) - ia_1 \Delta y_1(x) = f_1(x), \quad x \in \Omega_1, \tag{2.33}$$
$$\lambda y_2(x) - ia_2 \Delta y_2(x) = f_2(x), \quad x \in \Omega_2. \tag{2.34}$$

We solve (2.33), (2.34) for the case where $\tau'(t) = 0$, noting that the case where $\tau') \neq 0$ can be addressed similarly. Let $(\varphi_1, \varphi_2) \in \mathcal{V}$. Then, multiplying (2.33) (resp. (2.34) by φ_1 (resp. by φ_2) and integrating formally in Ω_1(resp. in Ω_2), we obtain after using (2.12)

$$\lambda a_1 \int_{\Omega_1} \nabla y_1(x).\nabla \overline{\varphi}_1(x)dx - ia_1^2 \int_{\Gamma_0} \Delta y_1(x).\frac{\partial \overline{\varphi}_1(x)}{\partial \nu} d\Gamma + ia_1^2 \int_{\Omega_1} \Delta y_1(x)\Delta \overline{\varphi}_1(x) d\Gamma = a_1 \int_{\Omega_1} \nabla f_1(x).\nabla \overline{\varphi}_1(x)dx. \tag{2.35}$$

$$\lambda a_2 \int_{\Omega_2} \nabla y_2(x).\nabla \overline{\varphi}_2(x)dx - ia_2^2 \int_{\Gamma_2} \Delta y_2(x)\frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma + ia_2^2 \int_{\Gamma_0} \Delta y_2(x).\frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma + ia_2^2 \int_{\Omega_2} \Delta y_2(x)\Delta \overline{\varphi}_2(x) d\Gamma = a_2 \int_{\Omega_2} \nabla f_2(x).\nabla \overline{\varphi}_2(x)dx. \tag{2.36}$$

We have by (2.32), (2.28) and (2.12),

$$-ia_2 \Delta y_2 = \frac{1}{(\alpha + \beta e^{-\lambda \tau(t)})} \frac{\partial y_2(x)}{\partial \nu} - \frac{1}{(\alpha + \beta e^{-\lambda \tau(t)})} v(x). \tag{2.37}$$

Inserting (2.37) into (2.36), gives

$$\lambda a_2 \int_{\Omega_2} \nabla y_2(x).\nabla \overline{\varphi}_2(x)dx + \frac{a_2}{(\alpha + \beta e^{-\lambda \tau(t)})} \int_{\Gamma_2} \frac{\partial y_2(x)}{\partial \nu} \frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma + ia_2^2 \int_{\Gamma_0} \Delta y_2(x).\frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma + ia_2^2 \int_{\Omega_2} \Delta y_2(x)\Delta \overline{\varphi}_2(x) d\Gamma = a_2 \int_{\Omega_2} \nabla f_2(x).\nabla \overline{\varphi}_2(x)dx + \frac{a_2}{(\alpha + \beta e^{-\lambda \tau(t)})} \int_{\Gamma_2} z_0(x) \frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma. \tag{2.38}$$

Summing up (2.35) and (2.38) yields

$$\Lambda((y_1, y_2), (\varphi_1, \varphi_2)) = \mathcal{F}(\varphi_1, \varphi_2), \tag{2.39}$$

where

$$\Lambda((y_1, y_2), (\varphi_1, \varphi_2)) = \lambda a_1 \int_{\Omega_1} \nabla y_1(x).\nabla \overline{\varphi}_1(x)dx + \lambda a_2 \int_{\Omega_2} \nabla y_2(x).\nabla \overline{\varphi}_2(x)dx + ia_1^2 \int_{\Omega_1} \Delta y_1(x)\Delta \overline{\varphi}_1(x) d\Gamma + ia_2^2 \int_{\Omega_2} \Delta y_2(x)\Delta \overline{\varphi}_2(x) d\Gamma + \frac{a_2}{(\alpha + \beta e^{-\lambda \tau(t)})} \int_{\Gamma_2} \frac{\partial y_2(x)}{\partial \nu} \frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma - ia_1^2 \int_{\Gamma_0} \Delta y_1(x).\frac{\partial \overline{\varphi}_1(x)}{\partial \nu} d\Gamma + ia_2^2 \int_{\Gamma_0} \Delta y_2(x).\frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma. \tag{2.40}$$

and $\mathcal{F} : \mathcal{V} \to \mathbb{C}$ is the linear form defined by

$$\mathcal{F}(\varphi_1, \varphi_2) = a_1 \int_{\Omega_1} \nabla f_1(x).\nabla \overline{\varphi}_1(x)dx + a_2 \int_{\Omega_2} \nabla f_2(x).\nabla \overline{\varphi}_2(x)dx + \frac{a_2}{(\alpha + \beta e^{-\lambda \tau(t)})} \int_{\Gamma_2} z_0(x) \frac{\partial \overline{\varphi}_2(x)}{\partial \nu} d\Gamma.$$

We note that the bilinear form Λ is not continuous on $\mathcal{V}$ neither is $\mathcal{F}$. To overcome this difficulty, we adapt an idea of [3]. We introduce the space

$$\mathcal{Z} = \{(\varphi_1, \varphi_2) \in \mathcal{V} : \Delta \varphi_k \in L^2(\Omega_k), k = 1, 2, a_1 \frac{\partial \varphi_1}{\partial \nu} = a_2 \frac{\partial \varphi_2}{\partial \nu} \text{ on } \Gamma_0, \frac{\partial \varphi_2}{\partial \nu} \in L^2(\Gamma_2)\},$$

on which we define the inner product

$$\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle = a_1 \int_{\Omega_1} \nabla \varphi_1(x).\nabla \overline{\psi}_1 dx + a_2 \int_{\Omega_2} \nabla \varphi_2(x).\nabla \overline{\psi}_2 dx + a_1^2 \int_{\Omega_1} \Delta \varphi_1(x)\Delta \overline{\psi}_1(x) d\Gamma + a_2^2 \int_{\Omega_2} \Delta \varphi_2(x)\Delta \overline{\psi}_2(x) d\Gamma + \int_{\Gamma_2} \frac{\partial \varphi_2(x)}{\partial \nu} \frac{\partial \overline{\psi}_2(x)}{\partial \nu} d\Gamma.$$

Then $\mathcal{Z}$ is a Hilbert space.
Applying the Cauchy-Schwarz inequality to each inner product on the right-hand side of (2.40), we obtain

$$|\Lambda((y_1,y_2),(\varphi_1,\varphi_2))| \leq \lambda a_1 \|\nabla y_1\|_{L^2(\Omega_1)} \|\nabla\varphi_1\|_{L^2(\Omega_1)} + \lambda a_2 \|\nabla y_2\|_{L^2(\Omega_2)} \|\nabla\varphi_2\|_{L^2(\Omega_2)} + a_1^2 \|\Delta y_1\|_{L^2(\Omega_1)} \|\Delta\varphi_1\|_{L^2(\Omega_1)} + a_2^2 \|\Delta y_2\|_{L^2(\Omega_1)} \|\Delta\varphi_2\|_{L^2(\Omega_1)} + \frac{a_2}{\alpha+\beta e^{-\lambda\widehat{\tau}}} \left\|\frac{\partial y_2}{\partial\nu}\right\|_{L^2(\Gamma_2)} \left\|\frac{\partial\overline{\varphi}_2}{\partial\nu}\right\|_{L^2(\Gamma_2)}. \tag{2.41}$$

(2.41) implies $\Lambda(.,.)$ is continuous on $\mathcal{Z}$.

For the coercivity of Λ, observe that

$$\Lambda((y_1,y_2),(y_1,y_2)) = \lambda a_1 \|\nabla y_1\|^2_{L^2(\Omega_1)} + \lambda a_2 \|\nabla y_2\|^2_{L^2(\Omega_2)} + ia_1^2 \|\Delta y_1\|^2_{L^2(\Omega_1)} + ia_2^2 \|\Delta y_2\|^2_{L^2(\Omega_2)} + \frac{a_2}{\alpha+\beta e^{-\lambda\tau(t)})} \left\|\frac{\partial y_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2}).$$

Hence,

$$\begin{aligned} |\Lambda((y_1,y_2),(y_1,y_2))| &\geq \frac{1}{2}\{\lambda a_1 \|\nabla y_1\|^2_{L^2(\Omega_1)} + \lambda a_2 \|\nabla y_2\|^2_{L^2(\Omega_2)} + \frac{a_2}{\alpha+\beta e^{-\lambda\widehat{\tau}}} \left\|\frac{\partial y_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2)}\} + \\ &\quad \frac{1}{2}\{a_1^2 \|\Delta y_1\|^2_{L^2(\Omega_1)} + a_2^2 \|\Delta y_2\|^2_{L^2(\Omega_2)}\} \\ &\geq \frac{\sigma}{2}\{a_1 \|\nabla y_1\|^2_{L^2(\Omega_1)} + a_2 \|\nabla y_2\|^2_{L^2(\Omega_2)} + \left\|\frac{\partial y_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2)} + \\ &\quad a_1^2 \|\Delta y_1\|^2_{L^2(\Omega_1)} + a_2^2 \|\Delta y_2\|^2_{L^2(\Omega_2)}\}, \end{aligned}$$

where

$$\sigma = min\{1, \lambda, \frac{a_2}{\alpha+\beta e^{-\lambda\widehat{\tau}}}\}. \tag{2.42}$$

$\mathcal{F}$ is also continuous on Z. Therefore, we conclude from the Lax-Millgram Theorem (see [21], p. 344) that for all $\mathcal{F} \in \mathcal{Z}'$, where $\mathcal{Z}'$ is the dual of $\mathcal{Z}$, there exists a unique solution $(y_1,y_2) \in \mathcal{Z}$ to (2.39) for all $(\varphi_1,\varphi_2) \in \mathcal{Z}$. Since $\mathcal{V}' \subset \mathcal{Z}'$, then for all $\mathcal{F} \in \mathcal{V}'$, there exists a unique solution $(y_1,y_2) \in \mathcal{Z}$ to (2.39) for all $(\varphi_1,\varphi_2) \in \mathcal{Z}$.

Moreover, by restricting the variational forms (2.35) (resp. 2.38) to functions for which $\frac{\partial\varphi_1}{\partial\nu} = 0$ (resp. $\frac{\partial\varphi_2}{\partial\nu} = 0$), we obtain

$$\lambda y_1(x) - a_1\Delta y_1(x) = f_1(x), \qquad x \in \Omega_1, \tag{2.43}$$

$$\lambda y_2(x) - a_2\Delta y_2(x) = f_2(x), \qquad x \in \Omega_2, \tag{2.44}$$

from which we deduce that $(\Delta y_1, \Delta y_2) \in \mathcal{V}$ since $(y_1,y_2) \in \mathcal{V}$ and $(f_1,f_2) \in \mathcal{V}$.

We return to the variational form (2.39) after using some integrations by parts:

$$\begin{aligned} &\lambda a_1 \int_{\Omega_1} \nabla y_1(x).\nabla\overline{\varphi}_1(x)dx - ia_1^2 \int_{\Omega_1} \nabla\Delta y_1(x).\nabla\overline{\varphi}_1(x)dx \\ &+ \lambda a_2 \int_{\Omega_2} \nabla y_2(x).\nabla\overline{\varphi}_2(x)dx - ia_2^2 \int_{\Omega_2} \nabla\Delta y_2(x).\nabla\overline{\varphi}_2(x)dx + \\ &+ ia_2^2 \int_{\Gamma_2} \Delta y_2(x)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma + \frac{a_2}{\alpha+\beta e^{-\lambda\tau(t)}} \int_{\Gamma_2} \frac{\partial y_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma = \\ &a_1 \int_{\Omega_1} \nabla f_1(x).\nabla\overline{\varphi}_1(x)dx + a_2 \int_{\Omega_2} \nabla f_2(x).\nabla\overline{\varphi}_2(x)dx + \\ &\frac{a_2}{(\alpha+\beta e^{-\lambda\tau(t)})} \int_{\Gamma_2} z_0(x)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma. \end{aligned} \tag{2.45}$$

(2.45) together with (2.43) and (2.44), yields

$$\frac{a_2}{\alpha+\beta e^{-\lambda\tau(t)}}\int_{\Gamma_2}\frac{\partial y_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma=-\int_{\Gamma_2}ia_2^2\Delta y_2(x)+\frac{a_2}{(\alpha+\beta e^{-\lambda\tau(t)})}\int_{\Gamma_2}v(x)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma. \tag{2.46}$$

(2.46) implies that

$$\begin{aligned}\frac{\partial y_2(x)}{\partial\nu}&=-ia_2(\alpha+\beta e^{-\lambda\tau(t)})\Delta y_2(x)+v(x)\text{ for }x\in\Gamma_2,\\&=-\alpha z(x,0)-\beta z(x,0)\text{ for }x\in\Gamma_2.\end{aligned}$$

as desired and consequently $(y_1,y_2)\in D(A(t))$. and thus, $\lambda I-A(t)$ is onto for some $\lambda>0$ and for all $t>0$. This shows that $A(t)$ is maximal for each fixed t. □

Lemma 2.6. *There exist constants C and m independent of t such that for all $t\in[0,T]$, the semigroup $\{S_t(s)\}_{s\geq 0}$ generated by $\mathcal{L}(t)$ satisfies*

$$\|S_t(s)u\|_{\mathcal{H}}\leq Ce^{ms}\|u\|_{\mathcal{H}}, \tag{2.47}$$

for all $u\in\mathcal{H}$ and $s\geq 0$.

Proof. Let $\varphi=(y_1,y_2,z)\in D(A(0))$, then

$$\begin{aligned}\|\varphi\|_s^2&=a_1\int_{\Omega_1}|\nabla y_1(x)|^2dx+a_2\int_{\Omega_2}|\nabla y_2(x)|^2dx+\xi\tau(s)\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2d\rho d\Gamma,\\\|\varphi\|_r^2&=a_1\int_{\Omega_1}|\nabla y_1(x)|^2dx+a_2\int_{\Omega_2}|\nabla y_2(x)|^2dx+\xi\tau(r)\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2d\rho d\Gamma,\end{aligned}$$

and

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2}\leq 1+\frac{\tau(s)-\tau(r)}{\tau(r)}.$$

From the mean value theorem, we have

$$\tau(s)-\tau(r)=\tau'(a)(s-r),\quad\text{where }a\in(r,s),$$

and thus,

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2}\leq 1+\frac{|\tau'(a)|}{\tau(r)}|s-r|.$$

By (1.11), τ' is bounded and therefore,

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2}\leq 1+\frac{|\tau'(a)|}{\tau(r)}|s-r|\leq 1+\frac{d}{\hat{\tau}}|s-r|,$$

which gives

$$\|\varphi\|_s^2\leq e^{\frac{d}{\hat{\tau}}|s-r|}\|\varphi\|_r^2, \tag{2.48}$$

and the desired inequality (2.47) follows from (2.48) with $C=1$ and $m=\frac{d}{\hat{\tau}}$. □

Lemma 2.7. *For the operator $\widetilde{A}(t)$ we have*

$$\frac{d}{dt}\widetilde{A}(t)\in L_*^\infty([0,T],B(D(A(0)),\mathcal{H}).$$

Proof. We have

$$\frac{d}{dt}\widetilde{A}(t)=\frac{d}{dt}A(t)-\kappa'(t)I,$$

where

$$\kappa'(t)=\frac{\tau''(t)\tau'(t)}{2\tau(t)\sqrt{\tau'(t)^2+1}}-\frac{\tau'(t)\sqrt{\tau'(t)^2+1}}{2\tau(t)^2},$$

and

$$\frac{d}{dt}A(t) = (0, 0, \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1))}{\tau(t)^2})^T.$$

By (1.11) and (1.12), $k'(t)$ and $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1))}{\tau(t)^2}$ are bounded on $[0, T]$. Thus,

$$\frac{d}{dt}\widetilde{A}(t) \in L^{\infty}_{*}([0, T], B(D(A(0)), \mathcal{H}).$$

as desired. □

The main result of this section can now be stated.

Theorem 2.8. *For any initial datum $Y_0 \in D(A(0))$, problem (2.13) has a unique solution*

$$Y \in C([0, +\infty), D(A(0))) \cap C^1([0, +\infty), \mathcal{H}). \tag{2.49}$$

Proof. It follows from (2.16), Lemma 2.3, Proposition 2.2, Lemma 2.6, Lemma 2.7 that $\widetilde{A}(t)$ satisfies all the hypothesis of Theorem 2.1. Therefore, for any initial datum $Y_0 \in D(\widetilde{A}(0))$ problem (2.14) has a unique solution

$$\widetilde{Y} \in C([0, +\infty), D(\widetilde{A}(0))) \cap C^1([0, +\infty), \mathcal{H}). \tag{2.50}$$

and the desired conclusion follows from the equality $Y(t) = e^{\theta(t)}\widetilde{Y}(t)$. □

3 Proof of the exponential stability result

We proceed in several steps.

Step 1.

First, we show that the energy function defined by (1.14) is decreasing.

Proposition 3.1. *The energy corresponding to any regular solution of problem (1.3)-(1.9) is decreasing and there exists a positive constant K such that*

$$\frac{d}{dt}E(t) \leq -K\int_{\Gamma_2}\left\{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x, t-\tau(t))|^2\right\}d\Gamma,$$

where

$$K = \min\{a_2\alpha - \frac{a_2\beta}{2\sqrt{1-d}} - \frac{\xi}{2}, \frac{\xi(1-d}{2} - \frac{a_2\beta\sqrt{1-d}}{2}\}.$$

Proof. Differentiating $E(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = {} & a_1\Re\int_{\Omega_1}\nabla\overline{y}_1(x,t).\nabla\partial_t y_1(x,t)dx + a_2\Re\int_{\Omega_2}\nabla\overline{y}_2(x,t).\nabla\partial_t y_2(x,t)dx + \\ & \xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 \partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma + \\ & \frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2 d\rho d\Gamma. \end{aligned} \tag{3.1}$$

Applying Green's Theorem to the first two integrals on the right-hand side of (3.1), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = {} & a_1\Re\int_{\Gamma_1}\frac{\partial\overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma + a_1\Re\int_{\Gamma_0}\frac{\partial\overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma + \\ & a_2\Re\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma - a_2\Re\int_{\Gamma_0}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma + \\ & \xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 \partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma + \\ & \frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2 d\rho d\Gamma.\end{aligned}$$

Recalling the boundary conditions (1.5)-(1.6) and the transmission conditions (1.7)-(1.8), we get

$$\begin{aligned}\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2}|\partial_t y_2(x,t)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2}\partial_t\overline{y}_2(x,t-\tau)\partial_t y_2(x,t)d\Gamma+ \\ \xi\tau(t)\Re\int_{\Gamma_2}\int_0^1\partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma+ \\ \frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1|\partial_t y_2(x,t-\tau(t)\rho)|^2\,d\rho d\Gamma.\end{aligned}\tag{3.2}$$

Now we have,

$$\partial_\rho y(x,t-\tau(t)\rho) = -\tau(t)\partial_t y(x,t-\tau(t)\rho), \tag{3.3}$$

$$\partial_\rho^2 y(x,t-\tau(t)\rho) = \tau(t)^2\partial_t^2 y(x,t-\tau(t)\rho). \tag{3.4}$$

Therefore,

$$\begin{aligned}&\Re\int_0^1\partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma = \\ &-\frac{1}{\tau(t)^3}\Re\int_0^1\partial_\rho\overline{y}_2(x,t-\tau(t)\rho)\partial_\rho^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \\ &= -\frac{1}{2\tau(t)^3}\Re\int_0^1(1-\tau'(t)\rho)\frac{d}{d\rho}|\partial_\rho y_2(x,t-\tau(t)\rho)|^2\,d\rho d\Gamma \\ &= -\frac{1}{2\tau(t)^3}\Re[(1-\tau'(t)\rho)\,|\partial_\rho y_2(x,t-\tau(t)\rho)|^2]_0^1 - \frac{\tau'(t)}{2\tau(t)^3}\int_0^1|\partial_\rho y_2(x,t-\tau(t)\rho)|^2\,d\rho \\ &= \frac{1}{2\tau(t)^3}[|\partial_\rho y_2(x,t)|^2 - (1-\tau'(t))\,|\partial_\rho y_2(x,t-\tau(t))|^2] - \frac{\tau'(t)}{2\tau(t)^3}\int_0^1|\partial_\rho y_2(x,t-\tau(t)\rho)|^2\,d\rho \\ &= \frac{1}{2\tau(t)}[|\partial_t y_2(x,t)|^2 - (1-\tau'(t))\,|\partial_t y_2(x,t-\tau(t))|^2] - \frac{\tau'(t)}{2\tau(t)}\int_0^1|\partial_t y_2(x,t-\tau(t)\rho)|^2\,d\rho.\end{aligned}\tag{3.5}$$

Inserting (3.5) into (3.2) yields

$$\begin{aligned}\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2}|\partial_t y_2(x,t)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2}\partial_t\overline{y}_2(x,t-\tau(t))\partial_t y_2(x,t)d\Gamma+ \\ \frac{\xi}{2}\int_{\Gamma_2}[|\partial_t y_2(x,t)|^2 - (1-\tau'(t))\,|\partial_t y_2(x,t-\tau(t))|^2]d\Gamma,\end{aligned}$$

from which we obtain after using Cauchy-Schwarz's inequality

$$\begin{aligned}\frac{d}{dt}E(t) \le -a_2\alpha\int_{\Gamma_2}|\partial_t y_2(x,t)|^2\,d\Gamma + \frac{a_2\beta}{2\sqrt{1-d}}\int_{\Gamma_2}|\partial_t y_2(x,t)|^2\,d\Gamma+ \\ \frac{a_2\beta\sqrt{1-d}}{2}\int_{\Gamma_2}|\partial_t y_2(x,t-\tau(t))|^2\,d\Gamma - \frac{\xi(1-d)}{2}\int_{\Gamma_2}|\partial_t y_2(x,t-\tau(t))|^2\,d\Gamma+ \\ \frac{\xi}{2}\int_{\Gamma_2}|\partial_t y_2(x,t)|^2\,d\Gamma,\end{aligned}$$

and consequently

$$\frac{d}{dt}E(t) \le -K\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau(t))|^2\}d\Gamma,$$

with

$$K = \min\{a_2\alpha - \frac{a_2\beta}{2\sqrt{1-d}} - \frac{\xi}{2}, \frac{\xi(1-d)}{2} - \frac{a_2\beta\sqrt{1-d}}{2}\}.$$

Assumption (1.15) implies that the constant K is positive, which concludes the proof of Proposition 3.1. □

Step 2.
Inspired by [16], we introduce the Lyapunov functionnal

$$\mathbb{E}(t) = E(t) + \gamma\{\Im \int_{\Omega_1} y_1(x)m(x).\nabla y_1(x)dx + \Im \int_{\Omega_2} y_2(x)m(x).\nabla y_2(x)dx + \mathcal{E}(t)\},$$

where γ is a positive constant that will be chosen later and $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) = \xi\tau(t) \int_{\Gamma_2} \int_0^1 e^{-2\tau(t)\rho} \left|\partial_t y_2(x, t - \tau(t)\rho)\right|^2 d\rho\, d\Gamma. \tag{3.6}$$

Lemma 3.2. *For γ small enough, the functional $\mathbb{E}$ is equivalent to the energy E, that is there exist two positive constants μ_1 and μ_2 such that*

$$\mu_1\mathbb{E}(t) \leq E(t) \leq \mu_2\mathbb{E}(t).$$

Proof. Using Cauchy-Schwarz's, Young's and Poincaré's inequalities, we obtain

$$\left|\Im \int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx\right| \leq \frac{\mathcal{M}C_p}{a_1}\{\frac{a_1}{2}\int_{\Omega_1} \left|\nabla y_1(x,t)\right|^2 dx\}, \tag{3.7}$$

where

$$\mathcal{M} = sup|m(x)|, \text{for } x \in \overline{\Omega},$$

$$\int_{\Omega_1} |f(x)|^2 dx \leq \mathcal{C}_p, \forall f \in H^1_{\Gamma_0}(\Omega_1),$$

$$\left|\Im \int_{\Omega_1} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right| \leq \frac{\mathcal{M}}{2}\int_{\Omega_2} \{|y_2(x,t)|^2 + |\nabla y_2(x,t)|^2\}dx. \tag{3.8}$$

But

$$\begin{aligned}
\int_{\Omega_2} \{|y_2(x,t)|^2 + |\nabla y_2(x,t)|^2\}dx &\leq (1+c)\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + c\int_{\Gamma_0} |y_2(x,t)|^2 d\Gamma \\
&\leq (1+c)\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + c\int_{\Gamma_0} |y_1(x,t)|^2 d\Gamma \\
&\leq (1+c)\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + \mathcal{C}_{tr}c\int_{\Omega_1} |\nabla y_1(x,t)|^2 dx \\
&\leq 2\mathcal{C}a\{\frac{a_2}{2}\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + \frac{a_1}{2}\int_{\Omega_1} |\nabla y_1(x,t)|^2 dx\},
\end{aligned} \tag{3.9}$$

where $a = min\{a_1, a_2\}$, $\mathcal{C} = max\{1+c, \mathcal{C}_{tr}c\}$, the constant $\mathcal{C}_{tr}$ is given by the trace's inequality

$$\int_{\Gamma_0} |f(x)|^2 d\Gamma \leq \mathcal{C}_{tr}\int_{\Omega_1} |\nabla f(x)|^2 dx, \forall f \in H^1_{\Gamma_0}(\Omega_1),$$

and the constant c is defined by (see [15])

$$\int_{\Omega_2} |y_2(x,t)| \leq c\{\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + \int_{\Gamma_0} |y_1(x,t)|^2 d\Gamma\}.$$

Combining (3.7), (3.8) and (3.9) gives us

$$\begin{aligned}
&\left|\Im \int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx + \Im \int_{\Omega_2} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right| \leq \\
&\mathcal{M}(\frac{\mathcal{C}_p}{a_1} + \mathcal{C}_p a)\{\frac{a_2}{2}\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + \frac{a_1}{2}\int_{\Omega_1} |\nabla y_1(x,t)|^2 dx\}.
\end{aligned} \tag{3.10}$$

On the other hand, it follows from (1.10), that

$$\mathcal{E}(t) \leq \xi\tau(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x, t-\tau(t)\rho)|^2 d\rho d\Gamma,$$

which together with (3.10) implies

$$\mu_1 \mathbb{E}(t) \leq E(t),$$

where

$$\mu_1 = (max\{1, \mathcal{M}\gamma(\frac{\mathcal{C}_p}{a_1} + \mathcal{C}_p a), 2\})^{-1}.$$

Now from (1.10), we have

$$\mathcal{E}(t) \geq \xi\tau(t)e^{-2\widetilde{\tau}} \int_{\Gamma_2}\int_0^1 |\partial_t y_2(x, t-\tau(t)\rho)d\rho d\Gamma| \,,$$

and

$$\left|\Im \int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx + \Im \int_{\Omega_1} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right| \geq$$
$$-(\frac{\mathcal{M}\mathcal{C}_p}{a_1} + \mathcal{M}\mathcal{C}_p a)\{\frac{a_2}{2}\int_{\Omega_2} |\nabla y_2(x,t)|^2 dx + \frac{a_1}{2}\int_{\Omega_1} |\nabla y_1(x,t)|^2 dx\}.$$

Therefore, for γ small enough,

$$E(t) \leq \mu_2 \mathbb{E}(t),$$

where

$$\mu_2 = min\{1 - \gamma\mathcal{M}\mathcal{C}_p(\frac{1}{a_1} + a), 1 + 2e^{-2\widetilde{\tau}}\}. \tag{3.11}$$

□

Lemma 3.3. *For any regular solution of problem (1.3)-(1.9), there exist positive constants C_0 and C_1 such that*

$$\frac{d}{dt}\Psi(t) \leq -C_0 E_s(t) + C_1\{\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x, t-\tau(t))|^2\} d\Gamma\}, \tag{3.12}$$

where

$$\Psi(t) = \sum_{k=1}^{2} \Im \int_{\Omega_k} y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx,$$

and

$$E_s(t) = \sum_{k=1}^{2} a_k \int_{\Omega_k} |\nabla y_k(t,x)|^2 dx.$$

Proof. We have

$$\frac{d}{dt}\Psi(t) = \sum_{k=1}^{2} \Im \int_{\Omega_k} \{\partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)} + y_k(t,x)m(x).\nabla\overline{\partial_t y_k(t,x)}\}dx.$$

By using Green's theorem, we get after using the boundary condition (1.5)

$$\Im \int_{\Omega_1} y_1(t,x)m(x).\nabla\overline{\partial_t y_1(t,x)}dx = \Im \int_{\partial\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}m(x).\nu(x)d\Gamma -$$
$$\Im \int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}div\, m(x)dx - \Im \int_{\Omega_1} \overline{\partial_t y_1(t,x)}m(x).\nabla y_1(t,x)dx$$
$$= -\Im \int_{\Gamma_0} y_1(t,x)\overline{\partial_t y_1(t,x)}m(x).\nu(x)d\Gamma - n\Im \int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}dx +$$
$$\Im \int_{\Omega_1} \partial_t y_1(t,x)m(x).\nabla\overline{y_1(t,x)}dx, \tag{3.13}$$

and

$$\begin{aligned}
&\Im\int_{\Omega_2} y_2(t,x)m(x).\nabla\overline{\partial_t y_2(t,x)}dx = \Im\int_{\partial\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma- \\
&\Im\int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}div\, m(x)dx - \Im\int_{\Omega_2}\overline{\partial_t y_2(t,x)}m(x).\nabla y_2(t,x)dx \\
&= \Im\int_{\Gamma_0} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma + \Im\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma- \\
&n\Im\int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}dx + \Im\int_{\Omega_2}\partial_t y_2(t,x)m(x).\nabla\overline{y_2(t,x)}dx.
\end{aligned} \tag{3.14}$$

Summing up (3.13) and (3.14) and recalling the boundary conditions (1.7) and (1.8), we obtain

$$\begin{aligned}
&\sum_{k=1}^{2}\Im\int_{\Omega_k} y_k(t,x)m(x).\nabla\overline{\partial_t y_k(t,x)}dx = \Im\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma+ \\
&\sum_{k=1}^{2}\Im\int_{\Omega_k}\{-ny_k(t,x)\overline{\partial_t y_k(t,x)} + \partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)}\}dx.
\end{aligned}$$

On the other hand, using equation (1.3), we get

$$\begin{aligned}
&\Im\int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}dx = -a_1\Re\int_{\Omega_1} y_1(t,x)\Delta\overline{y_1(t,x)}dx \\
&= a_1\Re\int_{\Gamma_0}\overline{y_1(t,x)}\frac{\partial y_1(t,x)}{\partial\nu}d\Gamma + a_1\int_{\Omega_1}|\nabla y_1(t,x)|^2\,dx,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\Im\int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}dx = -a_2\Re\int_{\Omega_2} y_2(t,x)\Delta\overline{y_2(t,x)}dx \\
&= -a_2\Re\int_{\Gamma_0} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma - a_2\Re\int_{\Gamma_2} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma + a_2\int_{\Omega_2}|\nabla y_2(t,x)|^2\,dx \\
&= -a_2\Re\int_{\Gamma_0} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma + a_2\alpha\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t)}d\Gamma+ \\
&a_2\beta\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t-\tau(t))}d\Gamma + a_2\int_{\Omega_2}|\nabla y_2(t,x)|^2\,dx.
\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) and using the transmission conditions (1.7) and (1.8) gives

$$\begin{aligned}
&\sum_{k=1}^{2}\Im\int_{\Omega_k}\partial_t y_k(t,x)\overline{y_k(t,x)}dx = a_2\alpha\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t)}d\Gamma+ \\
&a_2\beta\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t-\tau(t))}d\Gamma + \sum_{k=1}^{2}a_k\int_{\Omega_k}|\nabla y_k(t,x)|^2\,dx.
\end{aligned}$$

We also have from (1.3)

$$\sum_{k=1}^{2}\Im\int_{\Omega_k}\partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx = \sum_{k=1}^{2}a_k\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx.$$

Thus,

$$\frac{d}{dt}\Psi(t) = \Im \int_{\Gamma_2} \partial_t y_2(t,x)\overline{y_2(t,x)}m(x).\nu(x)d\Gamma - na_2\alpha\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t)d\Gamma -$$
$$na_2\beta\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t-\tau(t))d\Gamma - \sum_{k=1}^{2} na_k\int_{\Omega_k}|\nabla y_k(t,x)|^2\,dx +$$
$$2\sum_{k=1}^{2} a_k\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx.$$

Now we have,

$$2\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx = 2\Re\int_{\partial\Omega_k}\frac{\partial y_k(x,t)}{\partial\nu}m(x).\nabla\overline{y_k(t,x)}d\Gamma + (n-2)\int_{\Omega_k}|\nabla y_k(t,x)|^2\,dx -$$
$$\int_{\partial\Omega_k}|\nabla y_k(t,x)|^2\,d\Gamma.$$

Specializing this identity to $k=1$ and $k=2$, we find

$$2\Re\int_{\Omega_1}\Delta y_1(t,x)m(x).\nabla\overline{y_1(t,x)}dx = \int_{\Gamma_1}|\nabla y_1(t,x)|^2\,m(x).\nu(x)d\Gamma -$$
$$2\Re\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}m(x).\nabla\overline{y_1(t,x)}d\Gamma + \int_{\Gamma_0}|\nabla y_1(t,x)|^2\,m(x).\nu(x)d\Gamma + (n-2)\int_{\Omega_1}|\nabla y_1(t,x)|^2\,dx,$$

$$2\Re\int_{\Omega_2}\Delta y_2(t,x)m(x).\nabla\overline{y_2(t,x)}dx = 2\Re\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma - \int_{\Gamma_2}|\nabla y_2(t,x)|^2\,d\Gamma +$$
$$2\Re\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma - \int_{\Gamma_0}|\nabla y_2(t,x)|^2\,d\Gamma + (n-2)\int_{\Omega_2}|\nabla y_2(t,x)|^2\,dx,$$

and consequently

$$2\sum_{k=1}^{2} a_k\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx = a_1\int_{\Gamma_1}|\nabla y_1(t,x)|^2\,m(x).\nu(x)d\Gamma -$$
$$2a_1\Re\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}m(x).\nabla\overline{y_1(t,x)}d\Gamma + a_1\int_{\Gamma_0}|\nabla y_1(t,x)|^2\,m(x).\nu(x)d\Gamma +$$
$$2a_2\Re\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma - a_2\int_{\Gamma_2}|\nabla y_2(t,x)|^2\,d\Gamma + 2a_2\Re\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma -$$
$$a_2\int_{\Gamma_0}|\nabla y_2(t,x)|^2\,d\Gamma + (n-2)\sum_{k=1}^{2} a_k\int_{\Omega_k}|\nabla y_k(t,x)|^2\,dx. \tag{3.17}$$

We conclude from the boundary condition (1.7) that

$$\nabla(y_2(x,t)-y_1(x,t)) = \frac{\partial(y_2(x,t)-y_1(x,t))}{\partial\nu}\nu(x) \quad \text{on } \Gamma_0\times(0,T),$$

then

$$|\nabla y_2(x,t)|^2 = |\nabla y_1(x,t)|^2 + \left|\frac{\partial y_2(x,t)}{\partial\nu}\right|^2 - \left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 \quad \text{on } \Gamma_0\times(0,T),$$

so after using the boundary condition (1.8), we have on $\Gamma_0\times(0,T)$,

$$2a_1\Re(\frac{\partial y_1(x,t)}{\partial\nu}m(x).\nabla\overline{y_1(x,t)}) - 2a_2\Re(\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(x,t)}) - a_1|\nabla y_1(x,t)|^2\,m(x).\nu(x) +$$
$$a_2|\nabla y_2(x,t)|^2\,mx).\nu(x) = (a_2-a_1)|\nabla y_1(x,t)|^2\,m(x).\nu(x) - \frac{(a_2-a_1)^2}{a_2}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 m(x).\nu(x). \tag{3.18}$$

Inserting (3.18) into (3.17) yields

$$2\sum_{k=1}^{2} a_k \Re \int_{\Omega_k} \Delta y_k(t,x) m(x).\nabla \overline{y_k(t,x)} dx = a_1 \int_{\Gamma_1} |\nabla y_1(t,x)|^2 m(x).\nu(x) d\Gamma +$$
$$2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} m(x).\nabla \overline{y_2(t,x)} d\Gamma - a_2 \int_{\Gamma_2} |\nabla y_2(t,x)|^2 m(x).\nu(x) d\Gamma$$
$$+ (n-2)\sum_{k=1}^{2} a_k \int_{\Omega_k} |\nabla y_k(t,x)|^2 dx + (a_1 - a_2) \int_{\Gamma_0} |\nabla y_1(x,t)|^2 m(x).\nu(x) d\Gamma +$$
$$\frac{(a_2-a_1)^2}{a_2} \int_{\Gamma_0} \left| \frac{\partial y_1(x,t)}{\partial \nu} \right|^2 m(x).\nu(x) d\Gamma. \tag{3.19}$$

From (3.19), and invoking assumption (1.1), we deduce that

$$\frac{d}{dt}\Psi(t) \leq \Im \int_{\Gamma_2} \partial_t y_2(t,x) \overline{y_2(t,x)} m(x).\nu(x) d\Gamma + 2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} m(x).\nabla \overline{y_2(t,x)} d\Gamma -$$
$$a_2 \delta \int_{\Gamma_2} |\nabla y_2(t,x)|^2 d\Gamma - n a_2 \alpha \Re \int_{\Gamma_2} \overline{y_2(t,x)} \partial_t y_2(x,t) d\Gamma - n a_2 \beta \Re \int_{\Gamma_2} \overline{y_2(t,x)} \partial_t y_2(x, t-\tau(t)) d\Gamma -$$
$$2\sum_{k=1}^{2} a_k \int_{\Omega_k} |\nabla y_k(t,x)|^2 dx. \tag{3.20}$$

For the first term on the right-hand side of (3.20), we use Young's inequality, trace theorem and Poincaré's inequality to get the following estimate

$$\left| \Im \int_{\Gamma_2} \partial_t y_2(t,x) \overline{y_2(t,x)} m(x).\nu(x) d\Gamma \right| \leq \frac{\mathcal{M}^2}{2\epsilon} \int_{\Gamma_2} |\partial_t y_2(t,x)|^2 d\Gamma + \frac{\epsilon}{2} \int_{\Gamma_2} |y_2(t,x)|^2 d\Gamma$$
$$\leq \frac{\mathcal{M}^2}{2\epsilon} \int_{\Gamma_2} |\partial_t y_2(t,x)|^2 d\Gamma + \frac{\mathcal{C}_{tr}\epsilon}{2} \int_{\Omega_2} |\nabla y_2(t,x)|^2 dx$$
$$\leq \frac{\mathcal{M}^2}{2\epsilon} \int_{\Gamma_2} |\partial_t y_2(t,x)|^2 d\Gamma + \frac{\mathcal{C}_{tr}\epsilon}{2a_2} a_2 \int_{\Omega_2} |\nabla y_2(t,x)|^2 dx + \frac{\epsilon}{2a_1} a_1 \int_{\Omega_1} |\nabla y_1(t,x)|^2 dx, \tag{3.21}$$

where ϵ is an arbitrary positive constant.

For the second term, we have

$$2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} m(x).\nabla \overline{y_2(t,x)} d\Gamma \leq \frac{a_2 \mathcal{M}^2}{\delta} \int_{\Gamma_2} \left| \frac{\partial y_2(x,t)}{\partial \nu} \right|^2 d\Gamma + a_2 \delta \int_{\Gamma_2} |\nabla y_2(t,x)|^2 d\Gamma. \tag{3.22}$$

For the forth and the fifth term, we have after using Young's inequality and the trace theorem,

$$\left| n a_2 \alpha \Re \int_{\Omega_2} \overline{y_2(t,x)} \partial_t y_2(x,t) d\Gamma \right| \leq \frac{n\alpha a_2}{2\epsilon} \int_{\Gamma_2} |\partial_t y_2(x,t)|^2 d\Gamma dt +$$
$$\frac{n\alpha a_2 \epsilon \mathcal{C}_{tr}}{2} \int_{\Omega_2} |\nabla y_2(x,t)|^2 d\Gamma dt, \tag{3.23}$$
$$\left| n a_2 \alpha \Re \int_{\Omega_2} \overline{y_2(t,x)} \partial_t y_2(x, t-\tau(t)) d\Gamma \right| \leq \frac{n\alpha a_2}{2} \int_{\Gamma_2} |\partial_t y_2(x,t)|^2 d\Gamma dt +$$
$$\frac{n\alpha a_2}{2} \int_{\Gamma_2} |\partial_t y_2(x, t-\tau(t))|^2 d\Gamma dt. \tag{3.24}$$

Inserting (3.21)-(3.24) into (3.20) and recalling the boundary condition (1.6), we obtain ϵ small enough

$$\frac{d}{dt}\Psi(t) \leq -C_0 \sum_{k=1}^{2} a_k \int_{\Omega_k} |\nabla y_k(t,x)|^2 dx + C_1 \{ \int_{\Gamma_2} \{ |\partial_t y_2(x,t)|^2 + |\partial_t y_2(x, t-\tau(t))|^2 \} d\Gamma \},$$

where

$$C_0 = 2 - (\frac{1}{a_2} + \frac{1}{a_1} + n\alpha)\mathcal{C}_{tr}\epsilon,$$
$$C_1 = \frac{\mathcal{M}^2 + n\alpha a_2}{2\epsilon} + \frac{2a_2\mathcal{M}^2}{\delta} + \frac{n\alpha a_2}{2}.$$

C_0 is positive for ϵ small enough.

Lemma 3.4. *For any regular solution of problem (1.3)-(1.9),*

$$\frac{d}{dt}\mathcal{E}(t) \le -2\mathcal{E}(t) + \xi \int_{\Gamma_2} |\partial_t y_2(x,t)|^2 \, d\Gamma.$$

Proof. Differentiating both sides of (3.6) yields

$$\frac{d}{dt}\mathcal{E}(t) = \xi\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\Gamma-$$
$$2\xi\tau(t)\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\rho\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma+$$
$$2\xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma. \tag{3.25}$$

We have from (3.3) and (3.4)

$$\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma$$
$$= -(\tau(t))^{-3}\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_\rho\overline{y_2(x,t-\tau(t)\rho)}\partial_\rho^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma$$
$$= -\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_\rho\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)d\rho d\Gamma$$
$$= -\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}[e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)]_0^1 d\Gamma+$$
$$\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1\{-\tau'(t)e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 - 2\tau(t)e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma$$
$$= -\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_\rho y_2(x,t-\tau(t))\right|^2(1-\tau'(t)) - \left|\partial_\rho y_2(x,t)\right|^2\}d\Gamma-$$
$$\frac{\tau'(t)}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-$$
$$(\tau(t))^{-2}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma,$$

and then

$$\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho =$$
$$-\frac{1}{2}(\tau(t))^{-1}\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t)) - \left|\partial_t y_2(x,t)\right|^2\}d\Gamma-$$
$$\frac{\tau'(t)}{2}(\tau(t))^{-1}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-$$
$$\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma. \tag{3.26}$$

Inserting (3.26) into (3.25) leads to

$$\frac{d}{dt}\mathcal{E}(t) = \xi\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma -$$
$$2\xi\tau(t)\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\rho\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma -$$
$$\xi\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t)) - \left|\partial_t y_2(x,t)\right|^2\}d\Gamma -$$
$$\tau'(t)\xi\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma -$$
$$2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)d\rho d\Gamma,$$

and so

$$\frac{d}{dt}\mathcal{E}(t) = -2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma -$$
$$\xi\int_{\Gamma_2} e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t))d\Gamma + \xi\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma.$$

which gives the desired estimate. □

Completion of the proof of Theorem 1.1.

From Proposition 3.1, Lemma 3.3 and 3.4, we have

$$\frac{d}{dt}\mathbb{E}(t) \leq -K\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2 + \left|\partial_t y_2(x,t-\tau(t))\right|^2\}d\Gamma +$$
$$\gamma\left\{-C_0E_s(t) + (C_1+\xi)\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2 + \left|\partial_t y_2(x,t-\tau(t))\right|^2\}d\Gamma - 2\mathcal{E}(t)\right\} \tag{3.27}$$

Then for $\gamma(C_1+\xi) < K$, we get from (3.27)

$$\frac{d}{dt}\mathbb{E}(t) \leq -\gamma C_0E_s(t) - 2\gamma\mathcal{E}(t).$$

On the other hand, from the assumption (1.10), on $\tau(t)$, we deduce that

$$\mathcal{E}(t) \geq \xi\tau(t)e^{-2\widetilde{\tau}}\int_{\Gamma_2}\int_0^1\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma.$$

Therefore,

$$\frac{d}{dt}\mathbb{E}(t) \leq -\gamma C_0E_s(t) - \xi\tau(t)e^{-2\widetilde{\tau}}\int_{\Gamma_2}\int_0^1\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma \leq -\min\{\gamma C_0, \frac{e^{-2\widetilde{\tau}}}{2}\}E(t) \leq -C\mathbb{E}(t),$$

where

$$C = \mu_1\min\{\gamma C_0, \frac{e^{-2\widetilde{\tau}}}{2}\}.$$

This implies

$$\mathbb{E}(t) \leq e^{-Ct}\mathbb{E}(0),$$

and consequently

$$E(t) \leq \frac{\mu_2}{\mu_1}e^{-Ct}E(0).$$

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DECLARATION

The authors declare no conflict of interest.

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